A proof of Price's law for the collapse of a self-gravitating scalar field

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Abstract

A well-known open problem in general relativity, dating back to 1972, has been to prove *Price's law* for an appropriate model of gravitational collapse. This law postulates inverse-power decay rates for the gravitational radiation flux on the event horizon and null infinity with respect to appropriately normalized advanced and retarded time coordinates. It is intimately related both to astrophysical observations of black holes and to the fate of observers who dare cross the event horizon. In this paper, we prove a well-defined (upper bound) formulation of Price's law for the collapse of a self-gravitating scalar field with spherically symmetric initial data. We also allow the presence of an additional gravitationally coupled Maxwell field. Our results are obtained by a new mathematical technique for understanding the long-time behavior of large data solutions to the resulting coupled non-linear hyperbolic system of p.d.e.'s in 2 independent variables. The technique is based on the interaction of the conformal geometry, the celebrated red-shift effect, and local energy conservation; we feel it may be relevant for the problem of non-linear stability of the Kerr solution. When combined with previous work of the first author concerning the internal structure of charged black holes, which *assumed* the validity of Price's law, our results can be applied to the strong cosmic censorship conjecture for the Einstein-Maxwell-real scalar field system with complete spacelike asymptotically flat spherically symmetric initial data. Under Christodoulou's C^0-formulation, the conjecture is proven to be false.

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1 Introduction

A central problem in general relativity is to understand the final state of the collapse of isolated self-gravitating systems. One of the most fascinating predictions of this theory is the formation of so-called *black holes*. The fundamental issue is then to investigate their structure, both from the point of view of far away observers, and of those who dare cross the "event horizon".

The problem of gravitational collapse can be formulated mathematically as the study of the initial value problem for the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu}$$

coupled to appriopriate matter equations, where the initial data are assumed to be "asymptotically flat". The unknown in the Einstein equations is a time-oriented 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$ known as *spacetime*, the points of which are commonly called *events*. We employ in what follows standard language and concepts of global Lorentzian geometry.[1] In these terms, the notion of a black hole can be understood roughly as follows: Asymptotic flatness and the causal geometry of g allow one to distinguish a certain subset of the spacetime $\mathcal{M}$, namely the set of those events in the causal past of events arbitrarily "far away". The black hole region of spacetime is precisely the above subset's complement in $\mathcal{M}$.[2] The black hole's past boundary, a null hypersurface in $\mathcal{M}$, is what is known as the *event horizon*.

Current physical understanding of the basic nature of black holes–in particular, considerations regarding their stability, as measured by outside observers–rests more on folklore than on rigorous results. Central to this folklore is a heuristic study of the problem of linearized stability[3] around the Schwarzschild solution, carried out by Price [37] in 1972. Price's heuristics suggested that solutions of the wave equation on this background decay polynomially with respect to the static coordinate along timelike surfaces of constant area-radius. Later, these heuristics were refined [28] to suggest polynomial decay on the event horizon[4] itself, as well as along null infinity. In the context of the non-linear theory, Price's heuristics were interpreted to suggest that black holes with regular event horizons are stable from the point of view of far away observers and can form dynamically in collapse; moreover, that all "parameters" of the exterior spacetime other than mass, charge, and angular momentum–so-called "hair"–should

[1] In the effort of making this paper as self-contained as possible, all notions referred to in what follows are reviewed in a series of Appendices. Standard references in the subject are given at the end of this Introduction.

[2] In the context of this paper, we will be able to make this rigorous by defining first the concept of *null infinity*. One way to think about this is as a subset $\mathcal{I}^+$ of an appropriately defined "boundary" of $\mathcal{M}$, for which causal relations can still be applied. The black hole is then $\mathcal{M} \setminus J^-(\mathcal{I}^+)$. We call $J^-(\mathcal{I}^+)$ the *domain of outer communications* or more simply the *exterior*.

[3] In the linearized context, one can see already the most basic phenomena by considering the scalar wave equation $\Box_g \phi = 0$ on a fixed background g.

[4] Here, decay is measured with respect to an appropriately defined advanced time.

decay polynomially when measured along the event horizon or null infinity. The conjectured generic decay rates are now widely known as *Price's law*.

In this paper, we shall show that Price's law, as an upper bound for the rate of decay, indeed holds in the context of a suitable mathematical theory of gravitational collapse. The theory will be described by the Einstein-Maxwell-real scalar field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu}, \tag{1}$$

$$F_{\mu\nu}^{;\mu} = 0, F_{[\mu\nu,\lambda]} = 0, \tag{2}$$

$$g^{\mu\nu}\phi_{;\mu\nu} = 0, \tag{3}$$

$$T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi^{;\alpha}\phi_{;\alpha} + F_{\mu\alpha}F^{\alpha}_{\ \nu} - \frac{1}{4}g_{\mu\nu}F^{\alpha}_{\ \lambda}F^{\lambda}_{\ \alpha}. \tag{4}$$

We will assume our initial data to be spherically symmetric[5]. Physically, the above equations correspond to a self-gravitating system comprising of a neutral scalar field and an electromagnetic field. The motivation for this system has been discussed in previous work of the first author [23, 24]. Basically, (1)–(4) is the simplest self-gravitating system that radiates to infinity under spherical symmetry[6], and in addition has a repulsive mechanism (provided by the charge) competing with the attractive force of gravity. It can thus be thought of as a good first attempt at a spherically symmetrical model of the collapse of the vacuum equations *without* symmetry, where gravitational radiation and the centrifugal effects of angular momentum play an important role.[7]

We shall give a precise statement of the main theorem of this paper in Section 1.1 below. We then turn in Section 1.2 to a discussion of the *mass inflation scenario*, first proposed in 1990 by Israel and Poisson. According to this scenario, based on heuristic calculations, Price's law has implications for the internal structure of black holes, and the validity of the strong cosmic censorship conjecture. In view of previous results of the first author [24], we shall deduce from Theorem 1.1 that a suitable formulation of strong cosmic censorship is *false* for (1)–(4) under spherical symmetry (Corollary 1.1.2). In Section 1.3, we shall give a precise statement (Theorem 1.2) of Price's law in the linearized setting, for which our results also apply. Finally, we provide an outline for the body of the paper in Section 1.4, and in addition, point out some useful references.

[5]See Appendix B for a geometric definition. This assumption will imply that the system (1)–(4) reduces to a system in 2 spacetime dimensions.

[6]Recall, that in view of Birkhoff's theorem, the vacuum equations $R_{\mu\nu} = 0$ have no dynamical degrees of freedom in spherical symmetry.

[7]See, however, the remarks at the end of Section 1.2.

1.1 The main theorem

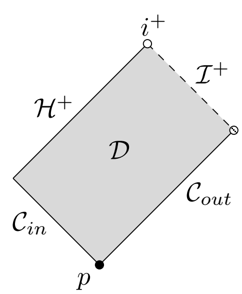
Price's law is a statement about precise decay rates in the exterior of a black hole in the Cauchy development of an appropriate initial data set. Because our system is self-gravitating, i.e. the spacetime $(\mathcal{M}, g)$ is itself unknown *a priori*, it follows that before one can talk about decay, one must show that indeed a black hole exists in the spacetime, identify its event horizon, and define time parameters in the exterior with respect to which decay is to be measured. All this must thus be contained in the statement of our theorem. To describe concisely the global geometry of spacetime, it will be convenient to employ so-called *Penrose diagrams*. This notation and related conventions are described in detail in Appendix C. It is an easy exercise to translate geometric statements exhibited by Penrose diagrams to statements about the existence and properties of certain null coordinate systems.[8]

The main result of this paper is:

Theorem 1.1. *Consider spacelike spherically symmetric initial data for the Einstein-Maxwell-scalar field equations, with at least one asymptotically flat end, such that the scalar field and its gradient have compact support on the initial hypersurface. Let $\mathcal{Q}$ denote the 2-dimensional Lorentzian quotient of the future Cauchy development. Assume either*

1. *$\mathcal{Q}$ contains a trapped surface,* or
2. *The initial hypersurface is complete, and the Maxwell field does not identically vanish,* or
3. *The initial hypersurface is complete and has two asymptotically flat ends.*[9]

Then, $\mathcal{Q}$ contains a subset $\mathcal{D}$ with Penrose diagram:



Let r denote the area-radius function, m the Hawking mass function and e

[8]For the convenience of the reader uncomfortable with diagrams, starting from Section 2, we will always supplement information inferred from diagrams with explicit formulas in terms of a related null coordinate system. In particular, the Penrose diagram of Theorem 1.1 is "translated" in $\mathbf{A}'$ of Section 2.

[9]It turns out that assumption 2 implies 3.

the charge[10]*. One can define a regular null coordinate system* (u, v) *covering* $\mathcal{D} \cap J^-(\mathcal{I}^+)$ *by setting* $p = (1, 1)$*, and defining the* v*-coordinate such that* $v \sim r$ *along the outgoing null ray* $\mathcal{C}_{out}$*, while defining the* u*-coordinate such that* $\partial_u r = -1$ *(in a suitable limiting sense) along null infinity* $\mathcal{I}^+$*. With this normalization,* $u \to \infty$ *along* $\mathcal{I}^+$ *as* i^+ *is approached, and, moreover, null infinity is complete in the sense of Christodoulou [13]. The points of the event horizon* $\mathcal{H}^+$ *can be parametrized by* (∞, v)*, and* $\mathcal{I}^+$ *can be parametrized by* (u, ∞)*. On* $\mathcal{H}^+$*, the quantities* $m(\infty, v)$*,* $r(\infty, v)$*,* $\phi(\infty, v)$*, and* $\partial_v\phi(\infty, v)$ *are defined, and* $r\phi(u, \infty)$ *and* $\partial_u(r\phi)(u, \infty)$ *can be defined on* $\mathcal{I}^+$ *by a limiting procedure. It follows that either "the black hole is extremal in the limit", i.e.*

$$m(\infty, v) + \frac{e^2}{2r(\infty, v)} - |e| \to 0 \text{ as } v \to \infty, \tag{5}$$

along $\mathcal{H}^+$*, or the following hold:*

$$|\partial_v r|(\infty, v) \geq C\left(1 - \frac{2m(\infty, v)}{r(\infty, v)}\right), \tag{6}$$

$$|\phi|(\infty, v) + |\partial_v\phi|(\infty, v) \leq C_\epsilon v^{-3+\epsilon}, \tag{7}$$

along $\mathcal{H}^+$*, and*

$$|r\phi|(u, \infty) \leq C_\epsilon u^{-2}, \tag{8}$$

$$|\partial_u(r\phi)|(u, \infty) \leq C_\epsilon u^{-3+\epsilon} \tag{9}$$

along $\mathcal{I}^+$*, for any* $\epsilon > 0$*, and for some positive constants* C*,* C_ϵ*.*

One should note that the system (1)–(4) has previously been studied numerically [8, 29, 33], again under spherical symmetry; Theorem 1.1 is in complete agreement with the findings of these studies.

In the case where $e = 0$, Christodoulou has shown [14] that for generic[11] initial data, either the solution disperses of a black hole forms. In the latter case, it is easy to deduce the assumptions of Theorem 1.1. We thus have

Corollary 1.1.1. *If* $e = 0$*, generic complete asymptotically flat BV-data in the sense of Christodoulou either have a future causally geodesically complete maximal development, or the result of Theorem 1.1 applies.*[12]

[10] Under spherical symmetry, the Maxwell part decouples, and its contribution to the evolution of the metric and the scalar field depends only on a constant–denoted e and called *charge*–which can be computed from initial data. We have $e = 0$ iff the Maxwell tensor $F_{\mu\nu}$ vanishes identically. See Appendix B.

[11] Here, the notion of genericity is defined in the context of a space of functions of bounded variation, which in turn is defined directly in terms of the spherically symmetric reduction. We refer to [18] for details.

[12] Previously, Christodoulou had shown in this case that $\phi(\infty, v) \to 0$ and $r\phi(u, \infty) \to 0$, without, however, a rate.

The global features of (1)–(4) with $e \neq 0$ in the *interior* of black hole regions turn out to be completely different from the case were $e = 0$. This is already clear, for example, by comparing the Penrose diagrams of the Schwarzschild and Reissner-Nordström solutions. (See Appendix C.4.) It is the existence of a smooth Cauchy horizon in the latter solutions that raises the question of strong cosmic censorship. We will turn in the next section to a discussion of this issue. The story turns out to be much more interesting than what one could conjecture from the examination of special solutions; moreover, the validity of Price's law plays a central role.

1.2 Mass inflation

The most basic mathematical questions regarding the internal structure of black holes concern the presence and nature of singularities. The old story goes that "deep inside" the black hole, spacetime should terminate in a singularity, strong enough to eventually destroy any observer who crosses the event horizon. The classic example is Schwarzschild (see Appendix C.4). The set labelled $\mathcal{B}$ on the boundary indeed represents a "singularity"; the spacetime is *inextendible* as a C^0 metric. The statement of strong cosmic censorship (see Appendix A.5) thus holds for this particular solution.

The Schwarzschild family of solutions can be considered, however, as a very special case of the Reissner-Nordström family of solutions of the Einstein-Maxwell system, for which the charge e happens to vanish. In the case $e \neq 0$, the situation is entirely different (see Appendix C.4). The spacetime is future causally geodesically incomplete, but there is no "singularity"; the Cauchy development $(\mathcal{M}, g)$ can be extended to a larger C^∞ spacetime $(\tilde{\mathcal{M}}, \tilde{g})$, still satisfying (1)–(4), such that all inextendible causal curves in $\mathcal{M}$ enter $\tilde{\mathcal{M}} \setminus \mathcal{M}$. The statement of strong cosmic censorship fails thus for this solution. The boundary of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ is known as the *Cauchy horizon*.[13]

The basis for the strong cosmic censorship conjecture were heuristic arguments, originally due to Penrose, which indicated that the Reissner-Nordström Cauchy horizon should be unstable. These arguments were put forth in the context of the linear theory and made use of the so-called *blue-shift effect*.[14] (See Hawking and Ellis [30], p. 161, for a discussion.) In 1990, Israel and Poisson took a closer look at this effect in a non-linear context motivated by the system (1)–(4). The picture they obtained was more complex than Penrose's original expectations, and, in particular, its implications for cosmic censorship turned out to be mixed.

To understand the strength of the blue-shift effect in the non-linear setting, one needs a quantitative estimate on the strength of infalling radiation into the black hole. But this is precisely the rate of decay of radiation tails on the event horizon, i.e. Price's law. On the one hand, Israel and Poisson calculated–in the

[13] In the Penrose diagram of the quotient of such an extension, this boundary would correspond to the set labelled $\mathcal{CH}^+$.

[14] This effect is in some sense dual to the *red-shift effect* which will play such an important role in the analysis in this paper.

context of their heuristics–that Price's power-law decay was in fact sufficiently *fast* so as to ensure Cauchy horizons still arise in the black hole interior, beyond which the spacetime metric can be *continuously* extended. On the other hand, they calculated, provided that the rate postulated by Price is sharp, in the sense that it provides–generically–a *lower* bound on the rate of decay, that it is sufficiently *slow* to ensure that these Cauchy horizons would generically be weakly singular, in particular the Hawking mass would blow up identically, and thus, the above-mentioned C^0 metric extensions could in fact not be C^1. In view of the blow-up of the mass, Israel and Poisson dubbed this scenario *mass inflation*.

In [24], the first author proved that under the assumption that a slightly weaker version of Price's law holds, the mass-inflation scenario of Israel and Poisson is correct for spherically-symmetric solutions of (1)–(4) with $e \neq 0$. In particular, it is assumed in [24] that an event horizon has formed, and that that the resulting black hole is non-extremal. The precise polynomial decay assumption imposed in [24] that guaranteed continuous extendibility of the metric was that

$$|\partial_v \phi| \leq C v^{-1-\epsilon} \tag{10}$$

along the event horizon[15], where ϵ is an arbitrarily small positive number and v a suitably normalized advanced time coordinate.

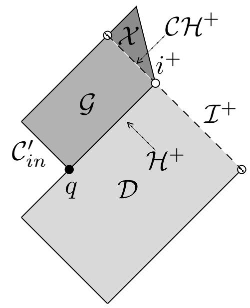
In fact, the condition (6) ensures that the v-coordinate described in Theorem 1.1 is equivalent to the v-coordinate employed in the formulation of [24]. Inequality (7) is then seen to be stronger than the necessary condition (10). It follows that, under the assumptions of Theorem 1.1, if $e \neq 0$ and (5) does not hold[16], then $\mathcal{Q}$ contains a short outgoing null ray $\mathcal{C}'_{in}$ emanating from a point q sufficiently close to i^+ (see the Penrose diagram below), so that our solution restricted to $(\mathcal{H}^+ \cap J^+(q)) \cup \mathcal{C}'_{in}$ satisfies the assumptions on characteristic initial data of Theorem 1.1 of [24]. Thus, we have

Corollary 1.1.2. *Under the assumptions of the previous theorem, if in addition $e \neq 0$, then either (5) holds on $\mathcal{H}^+$, or the Cauchy development $(\mathcal{M}, g)$ is extendible to a larger manifold with C^0 metric. In particular, the Lorentzian quotient $\mathcal{Q}$ contains a subset with Penrose diagram depicted below as the union*

[15] The even weaker assumption $|\partial_v \phi| \leq C v^{-\frac{1}{2}-\epsilon}$ was sufficient to show that the area radius remained bounded away from 0. The results of [24] improved previous work [23] of the first author, where the mass-inflation scenario was shown to hold provided that the scalar field is compactly supported on the event horizon.

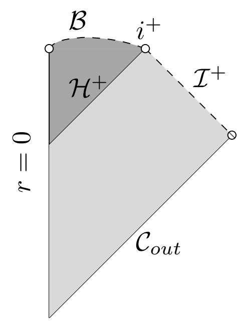
[16] It is easy to identify large open classes of initial data such that (5) does not occur: For instance, if $r > e$ everwhere on the initial hypersurface.

$\mathcal{D} \cup \mathcal{G}$:



The extension $\tilde{\mathcal{M}}$ can be taken spherically symmetric with quotient manifold $\tilde{\mathcal{Q}}$ extending globally "across" $\mathcal{CH}^+$, i.e. such that $\tilde{\mathcal{Q}}$ contains a subset $\mathcal{D} \cup \mathcal{G} \cup \mathcal{CH}^+ \cup \mathcal{X}$ depicted in the Penrose diagram above. It follows that the strong cosmic censorship conjecture, under the formulation[17] *of Christodoulou [16], is* false *in the context of spherically symmetric solutions of this system.*

To understand the significance of the above it is instructive to compare with the case $e = 0$. In [19], Christodoulou showed that if a black hole forms[18], then the 2-dimensional quotient $\mathcal{Q}$ has Penrose diagram:



The future boundary $\mathcal{B}$ of the black hole region $\mathcal{Q} \setminus J^-(\mathcal{I}^+)$ is such that the spacetime is locally inextendible across it as a C^0 metric. In particular, strong cosmic censorship is *true* in this case.

Finally, in connection to the system (1)–(4), it is important, to point out the following: While modelling the important physical effect of the centrifugal force of angular momentum, our system introduces another feature, which is entirely foreign to the physical problem. If $e \neq 0$, the equations do not admit complete spherically symmetric asymptotically flat initial data with one end.[19] In particular, the black holes to which our theorem applies are generated by the topology of the initial data. A model that would allow for *both* realistic

[17] See Appendix A.2. In this formulation, inextendibility is required in C^0.

[18] Recall that, as discussed above, in the case $e = 0$ it is proven in [14] that for generic initial data, either a black hole forms or the solution is future causally geodesically complete.

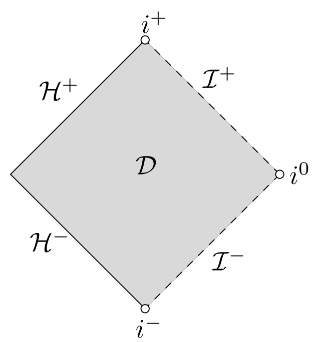
[19] This arises from the fact that (2) and (3) are not coupled directly. The source for Maxwell tensor must thus be topological.

initial data *and* the possibility of the formation of regular or partially regular Cauchy horizons is provided by the charged scalar field, discussed in Section 3.3 of Hawking and Ellis [30]. Unfortunately, at present there is neither a theory of the formation of black holes for this system of equations, in the spirit of [14], nor the internal causal structure of these black holes, in the spirit of [24]. The local existence theory for this system and small data results, however, have been proven (under spherical symmetry) by Chae [9].

1.3 The linearized problem

The method of this paper can in fact be specialized to provide a proof of decay in the uncoupled problem $\Box_g \phi = 0$, where g is Schwarzschild or Reissner-Nordström, heuristically studied in [37, 28, 4]. We obtain

Theorem 1.2. *Fix real numbers $0 \leq |e| < M$, and consider the standard maximally analytic Reissner-Nordström solution $(\mathcal{M}, g_{RN})$, with parameters e, M. (See Hawking and Ellis [30], p. 158.) Let $\mathcal{Q}$ denote the Lorentzian quotient $\mathcal{M}/SO(3)$ and π_1 the projection $\pi_1 : \mathcal{M} \to \mathcal{Q}$. Consider a subset $\mathcal{D} \subset \mathcal{Q}$ with Penrose diagram*[20]*:*



Let ϕ denote a C^2 solution to the wave equation on $\pi_1^{-1}(\mathcal{D})$, and define $\phi_0 : \mathcal{Q} \to \mathbf{R}$ by $\phi_0(p) = \frac{1}{4\pi r^2} \int_{\pi_1^{-1}(p)} \phi$.[21] *Let u and v denote standard Eddington-Finkelstein coordinates on $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-) \cap \mathcal{D}$, i.e. such that*

$$g_{RN} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) du dv + r^2 \gamma,$$

where γ is the standard metric on the 2-sphere. If u is such that constant-u curves are outgoing, then the future event horizon $\mathcal{H}^+$ thus corresponds to points (∞, v) and the past event horizon $\mathcal{H}^-$ to $(u, -\infty)$. Assume ϕ and $\nabla\phi$ satisfy $|r\phi| \leq \bar{C} r^{-2}$, $|\partial_u(r\phi)| + |\partial_v(r\phi)| \leq \bar{C} r^{-3}$ on a Cauchy surface $\mathcal{S} \subset \mathcal{D}$. Since the points (∞, v) and $(u, -\infty)$ are contained in $\mathcal{D}$, $\phi_0(\infty, v)$ and

[20] See Appendix C.2. In particular, note that according to the conventions described there, $\mathcal{H}^+ \cup \mathcal{H}^- \subset \mathcal{D}$

[21] This is the 0'th spherical harmonic. Recall that $p \in \mathcal{Q}$ means that $\pi_1^{-1}(p)$ represents a spacelike sphere in $\mathcal{M}$. The integral is with respect to the area element of $\pi_1^{-1}(p)$.

$\phi_0(u,-\infty)$ are clearly defined, and one can see that $\partial_v\phi_0(\infty,v)$, $\partial_u\phi_0(u,-\infty)$ are likewise defined. Moreover, in this coordinate system, points (u,∞), $(-\infty,v)$ can be identified with points of $\mathcal{I}^+$, $\mathcal{I}^-$, respectively, and $r\phi_0(u,\infty)$, $r\phi_0(-\infty,v)$, $\partial_u(r\phi_0)(u,\infty)$, $\partial_v(r\phi_0)(-\infty,v)$ can be defined. The following decay rates hold

$$|\partial_v\phi_0(\infty,v)|+|\phi_0(\infty,v)|\leq C_\epsilon(\max\{v,1\})^{-3+\epsilon},$$

$$|\partial_u\phi_0(u,-\infty)|+|\phi_0(u,-\infty)|\leq C_\epsilon(\max\{-u,1\})^{-3+\epsilon},$$
$$|r\phi_0|(u,\infty)\leq C|u|^{-2},$$
$$|r\phi_0|(-\infty,v)\leq C|v|^{-2},$$
$$|\partial_u(r\phi_0)|(u,\infty)\leq C|u|^{-3+\epsilon},$$
$$|\partial_v(r\phi_0)|(-\infty,v)\leq C|v|^{-3+\epsilon}.$$

In terms of (r,t) coordinates where t is the static time defined by $t=\frac{1}{2}(v+u)$, we have

$$|\partial_t\phi|(r_0,t)+|\partial_r\phi|(r_0,t)+|\phi|(r_0,t)\leq C_{r_0,\epsilon}|t|^{-3+\epsilon}$$

for each fixed $r_0>r_+=M+\sqrt{M^2-e^2}$.

What connects the linear and the non-linear problem, in such a way so as to allow the above "specialization", is the existence, in both cases, of an analogous energy estimate for ϕ. In the static case, this estimate is deduced by contracting the energy-momentum tensor of ϕ with the $\frac{\partial}{\partial t}$ Killing vector field, while in the coupled case, the estimate arises through the coupling, via the remarkable properties of the (renormalized) Hawking mass.

A proof of a weaker[22] version of this linearized Price's law has recently been given by Machedon and Stalker [32]. Other studies of this linear problem have appeared in the physics literature, e.g. [11]. The boundedness of the solution (without the assumption of spherical symmetry) was proven by Kay and Wald [31], while decay in t, without however a rate, was proven by Twainy [40].

The standard approach for the linearized problem has been to adopt the special Regge-Wheeler coordinates, which transform the wave equation on a static backround to the Minkowski-space wave equation with time-independent potential. Fourier analytic and/or spectral theoretic methods can then be employed. It would be interesting to investigate whether these methods would also be applicable to the non-linear problem studied here. The method of the present paper, however, is in a completely different, new direction. Here it is the global characteristic geometry of the event horizon, together with the celebrated red-shift effect, that will play the central role. Both these aspects seem to be somewhat obscured by the Regge-Wheeler coordinate approach.

[22] Decay rates are derived in the Schwarzschild t coordinate but do not extend to decay rates on the event horizon with respect to advanced time. Also, it is assumed that the solution vanishes in a neighborhood of $\mathcal{H}^+\cap\mathcal{H}^-$ and i^0.

1.4 Outline of the paper

A brief outline of the paper is as follows: In Section 2, we shall formulate the most general assumptions for which we can prove our results. These will refer to a 2-dimensional Lorentzian manifold $(\mathcal{Q}, \bar{g})$ on which two functions r, ϕ are defined, satisfying various properties postulated *a priori*. In Section 3, we retrieve the assumptions of Section 2 from various more recognizable assumptions, including (but not limited to) those of Theorem 1.1. In Section 4, we shall introduce the most basic analytic features of the system of equations formulated in Section 2. We shall then proceed to prove a uniform boundedness (and uniform decay in r) result in Section 5. In Sections 6–8, we prove certain localized estimates in characteristic rectangles. These estimates relate behavior on various boundary segments to behavior in the interior. Decay is then proven in Section 9, where the estimates of Sections 6–8 are applied to sequences of rectangles constructed with the help of the global causal geometry, energy conservation, and the pigeonhole principle. The final results are given by Theorems 9.4 and 9.5 of Section 9.3. We formulate some open problems in Section 10, and finally, we outline and the modifications of our argument necessary to prove Theorem 1.2, in Section 11.

As noted in the very beginning, we have included, for the benefit of the reader not so familiar with the standard language of the global study of the Einstein equations, several appendices. In Appendix A, we discuss the terminology necessary to understand the global initial value problem for the Einstein equations: basic Lorentzian geometry, the notion of the maximal Cauchy development, Penrose's singularity theorems, asymptotically flat initial data, and finally, the statement of the cosmic censorship conjectures. Whenever convenient, we will restrict consideration to the system (1)–(4). In Appendix B, we review the definition and basic facts about spherical symmetry, in particular the definition of quotient manifold $\mathcal{Q}$, area radius r, and Hawking mass m. Finally, in Appendix C, we introduce the notational convention of Penrose diagrams.

For more, the reader should consult standard textbooks, like Hawking and Ellis [30], or [3], as well as the nice survey article [16]. References [25, 27] might also be helpful.

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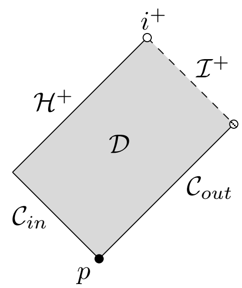
2 Assumptions

In this section, we will formulate the most general assumptions under which the results of our paper apply. These assumptions ($\mathbf{A}'$–$\mathbf{\Sigma T}'$ below) will refer to

a 2-dimensional spacetime $(\mathcal{Q}, \bar{g})$, on which two functions r and ϕ are defined, and a constant e. As we shall see, only weak regularity assumptions need be imposed.

A′. We assume we have a C^3 2-dimensional manifold $\mathcal{Q}$ on which a C^1 Lorentzian metric $\bar{g}$ and a C^2 nonnegative function r are defined, such that $\mathcal{Q}$ contains a subset $\mathcal{D}$ on which r is strictly positive, and such that $\mathcal{D}$ has Penrose diagram[23] depicted below:



Note that in view of our conventions of the Appendices, the above diagram encodes in particular the inclusions $\mathcal{H}^+ \subset \mathcal{D}$, $\mathcal{C}_{out} \subset \mathcal{D}$, $\mathcal{C}_{in} \subset \mathcal{D}$, while, again by our convention, $i^+ \notin \mathcal{I}^+$, and thus, by the definition of $\mathcal{I}^+$, $\sup_{\mathcal{H}^+} r = r_+ < \infty$.

For the reader uncomfortable with diagrams, we translate: There exists a global null coordinate chart (u, v) covering $\mathcal{D}$ such that

$$\bar{g} = -\frac{\Omega^2}{2}(du \otimes dv + dv \otimes du)$$

with Ω a strictly positive C^1 function. We assume that $\mathcal{D}$ corresponds precisely to the coordinate range $[1, U_0] \times [1, V_0)$ for some $1 < U_0 < \infty$, $1 < V_0 \leq \infty$, and moreover, that u and v increase towards the future, i.e. ∇u, and ∇v are future pointing. We define $\mathcal{C}_{out} \subset \mathcal{D}$ to be the constant-u curve $\{1\} \times [1, V_0)$, we define $\mathcal{C}_{in}$ to be the constant-v curve $[1, U_0] \times \{1\}$, we define $p = (1, 1)$, and we define $\mathcal{H}^+$ to be $\{U_0\} \times [1, V_0)$. We assume that for all $u \in [1, U_0)$, $\sup_v r(u, v) = \infty$, whereas we assume $\sup_v r(U_0, v) = r_+ < \infty$.

B′. We assume a C^1 real-valued function ϕ to be defined on $\mathcal{D}$.

Γ′. Let (u, v) be a null coordinate system as in **A′**, With respect to such coordinates define the functions ν, λ by

$$\partial_u r = \nu, \tag{11}$$

$$\partial_v r = \lambda. \tag{12}$$

[23]See Appendix C for an explanation of this notation, or else, skip to the next paragraph!

We assume

$$\lambda \geq 0 \tag{13}$$

and

$$\nu < 0. \tag{14}$$

On $J^-(\mathcal{I}^+) \cap \mathcal{D} = \mathcal{D} \setminus \mathcal{H}^+$ we assume

$$\lambda > 0. \tag{15}$$

Δ'. Define the functions θ, ζ by

$$r\partial_u \phi = \zeta \tag{16}$$

$$r\partial_v \phi = \theta. \tag{17}$$

Fix some real number ω such that $1 < \omega \leq 3$. On $\mathcal{C}_{out}$, assume[24]

$$r\left|\frac{\theta}{\lambda}\right| \leq C, \tag{18}$$

$$r^\omega \left|\phi + \frac{\theta}{\lambda}\right| \leq C, \tag{19}$$

for some constant C.

E′. Define the function m by

$$m = \frac{r}{2}(1 - \bar{g}(\nabla r, \nabla r)). \tag{20}$$

We will call m the *Hawking mass*. Define μ, the so-called *mass-aspect function*, by

$$\mu = \frac{2m}{r}. \tag{21}$$

Assume that

$$0 < \sup_{\mathcal{C}_{out}} m < \infty. \tag{22}$$

Fix a constant e, to be called the *charge*, and define the *renormalized Hawking mass* ϖ by

$$\varpi = m + \frac{e^2}{2r} = \frac{r}{2}(1 - \bar{g}(\nabla r, \nabla r)) + \frac{e^2}{2r}. \tag{23}$$

In view of (14), a function κ can be defined everywhere on $\mathcal{D}$ by

$$\kappa = -\frac{1}{4}\Omega^2 \nu^{-1}. \tag{24}$$

[24]Note that $\frac{\theta}{\lambda}$ is invariant with respect to reparametrization of the v coordinate.

By (20), we have the identity

$$\lambda = \kappa(1-\mu). \tag{25}$$

Assume that ϖ, κ, θ, ζ satisfy pointwise throughout $\mathcal{D}$ the equations

$$\partial_u \varpi = \frac{1}{2}(1-\mu)\left(\frac{\zeta}{\nu}\right)^2 \nu, \tag{26}$$

$$\partial_v \varpi = \frac{1}{2}\kappa^{-1}\theta^2, \tag{27}$$

$$\partial_u \kappa = \frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2 \nu\kappa \tag{28}$$

$$\partial_u \theta = -\frac{\zeta\lambda}{r}, \tag{29}$$

$$\partial_v \zeta = -\frac{\theta\nu}{r}. \tag{30}$$

(In particular, assume θ, ζ are differentiable in the u and v directions, respectively.)

$\mathbf{\Sigma T}'$. Assume $r_+ \neq |e|$.

The above assumptions are clearly independent of the choice of null coordinate system in $\mathbf{B}'$. That is to say, given a null coordinate system (u, v) covering $\mathcal{D}$ and functions $\nu, \lambda, \theta, \zeta, \kappa$ satisfying Assumptions $\mathbf{\Gamma}'$–$\mathbf{\Sigma T}'$, then if (u^*, v^*) is a new null coordinate system covering $\mathcal{D}$ with $\frac{\partial u}{\partial u^*}$, $\frac{\partial v}{\partial v^*}$ positive C^1 functions, defining $\nu^* = \frac{\partial r}{\partial u^*} = \frac{\partial u}{\partial u^*}\nu$, $\lambda^* = \frac{\partial r}{\partial v^*} = \frac{\partial u}{\partial u^*}\lambda$, etc., these quantities again satisfy Assumptions $\mathbf{\Gamma}'$–$\mathbf{\Sigma T}'$. Indeed, we have formulated the assumptions in the above manner precisely in order to emphasize their invariant geometric content. We defer till the next section the task of identifying a particular convenient null coordinate system.

From (25), (26), and (11) we obtain

$$\partial_u \lambda = \nu \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right). \tag{31}$$

By equality of mixed partials, we then have

$$\partial_v \nu = \nu \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right). \tag{32}$$

In addition, note that where $1-\mu \neq 0$, $\lambda \neq 0$, we have, in analogy to (28),

$$\partial_v \left(\frac{\nu}{1-\mu}\right) = \frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2 \lambda \left(\frac{\nu}{1-\mu}\right). \tag{33}$$

Finally, we note the following: When denoting causal relations in $\mathcal{D}$, *e.g.* $p \in J^+(q)$, it goes without saying that J^+, etc. will always refer to the causal structure of $\bar{g}$ on $\mathcal{D}$. It is clear, however, that, in any future pointing null coordinate system on $\mathcal{D}$, as in $\mathbf{A}'$, the causal structure of $\bar{g}$ is identical to the causal structure of the domain of the coordinates as a subset of 2-dimensional Minkowski space, i.e., the set $J^+(u,v)$, $J^-(u,v)$, etc., are given in null coordinates as in the formulas of Appendix C.1.

3 Retrieving the assumptions

In this section we give three alternative propositions all of which retrieve the assumptions $\mathbf{A}'$–$\mathbf{\Sigma T}'$ of the previous section.

In the first proposition, we retrieve $\mathbf{A}'$–$\mathbf{\Sigma T}'$ from the assumptions of Theorem 1.1:

Proposition 3.1. *Let $\{\Sigma, \overset{\circ}{g}, K_{ij}, E_i, B_i, \overset{\circ}{\phi}, \phi'\}$ be an asymptotically flat spherically symmetric initial data set*[25]*, such that $\overset{\circ}{\phi}$, ϕ' are of compact support. Let*

$$\{\mathcal{M}, g, F_{\mu\nu}, \phi\} \tag{34}$$

denote its future Cauchy development, and let $(\mathcal{Q}, \bar{g})$ be the quotient manifold of (34) with its induced Lorentzian metric. Suppose either

1. *$\mathcal{Q}$ contains a trapped surface,* or
2. *$(\Sigma, \overset{\circ}{g})$ is complete and E_i and B_i do not both identically vanish,* or
3. *$(\Sigma, \overset{\circ}{g})$ is complete with two ends.*

Then $\mathcal{Q}$ contains a subset $\mathcal{D}$, such that if r is defined by (197), $\phi : \mathcal{Q} \to \mathbf{R}$ denotes the induced map of $\phi : \mathcal{M} \to \mathbf{R}$ on group orbits, and e is defined by (198), then $\{\mathcal{Q}, \bar{g}, r, \phi, e\}$ satisfy Assumptions $\mathbf{A}'$–$\mathbf{E}'$ with $\omega = 3$. If, in addition, $e = 0$ or $r > e$ identically along $\mathcal{S} = \Sigma/SO(3)$, then Assumption $\mathbf{\Sigma T}'$ is also satisfied.

Proof. The equations (11)–(30) were derived from (1)–(4) in [23]. The global structure implicit in the Penrose diagram of $\mathbf{A}'$ follows from the results of [26]. $\square$

In the spherically symmetric $e = 0$ case considered by Christodoulou, we can formulate a different proposition by appealing to the results of [15]:

Theorem 3.1. *Consider the initial value problem of [18], where it is assumed that (18) and (19) hold initially, and let $\{\mathcal{Q}, \bar{g}, r, \phi\}$ denote the BV future Cauchy development. Assume $\mathcal{Q}$ possesses a complete*[26] *null infinity $\mathcal{I}^+$, and*

[25] See Appendix A.2.

[26] Complete can be taken in the sense of [16], or merely as the statement that the u-coordinate defined as in Section 5.3 satisfies $u \to \infty$.

that the final Bondi mass is strictly positive. Assume, moreover, that $\frac{\zeta}{\nu}$ is uniformly bounded on some ingoing null ray intersecting the event horizon $\mathcal{H}^+$.[27] *Then $\{\mathcal{Q}, \bar{g}, r, \phi\}$ satisfy Assumptions $\mathbf{A}'$–$\mathbf{\Sigma T}'$ with $e = 0$.*

We note that the existence of a trapped surface in the BV-Cauchy development implies that null infinity is complete and that the final Bondi mass is strictly positive (see for instance [26]). Thus, in the context of Christodoulou's initial value problem, Theorem 3.1 contains Proposition 3.1. It would be nice if the results of this paper could also be proven for the more weak general class of solutions considered in [15]. This remains, however, an open problem.

Finally, we note that in the original heuristic analysis of Price, besides data of compact support, data which have an "initially static" tail are also considered. For this we have the following:

Proposition 3.2. *In Proposition 3.1, if the assumption that $\overset{\circ}{\phi}$, ϕ' are of compact support is replaced by the assumption that K_{ij}, ϕ' are of compact support, then Assumptions $\mathbf{A}'$–$\mathbf{\Sigma T}'$ hold as before, with $\omega = 2$.*

Our final conclusions (Theorems 9.4 and 9.5) will depend on ω in complete agreement with the heuristics of Price.

4 Basic properties of the system

In the rest of the paper, we assume always $\mathbf{A}'$–$\mathbf{\Sigma T}'$. Before proceeding into the details of our argument, we make some introductory comments about the structure of the system (11)–(30).

4.1 Coordinates

To get started, let us settle for the time being on a definite coordinate system. (We will change it soon enough, but no matter!) First note that by (15) and (25) it follows that

$$1 - \mu > 0$$

on $\mathcal{C}_{out}$. Since r is strictly increasing to the future on $\mathcal{C}_{out}$, and since[28] $\sup_{\mathcal{C}_{out}} r = \infty$, it follows that the set $\mathcal{C}_{out} \cap \{r \leq 3 \sup_{\mathcal{C}_{out}} m\}$ is compact. Thus $1 - \mu$ is bounded above and below by positive constants on this set, while $\frac{5}{3} \geq 1 - \mu \geq \frac{1}{3}$ on $\mathcal{C}_{out} \cap \{r \geq 3 \sup_{\mathcal{C}_{out}} m\}$. It follows that

$$\tilde{d} > 1 - \mu > d > 0$$

on $\mathcal{C}_{out}$ for some constants d, $\tilde{d}$.

[27] See Appendix C.2 for an explanation of this notation.

[28] If the reader wishes to interpret directly the Penrose diagram, note that by our conventions, $\mathcal{C}_{out}$, as depicted, has a limit point on $\mathcal{I}^+$!

We now set p by $p = (1,1)$, and then require

$$\frac{\partial_v r}{1-\mu} = \kappa = 1 \tag{35}$$

along $\mathcal{C}_{out}$, and

$$\nu = \partial_u r = -1$$

along $\mathcal{C}_{in}$. This clearly completely determines a null regular coordinate system in $\mathcal{D}$. To examine its range, note first that since $1-\mu < \tilde{d}$ on $\mathcal{C}_{out}$, we have, for $q \in \mathcal{C}_{out}$

$$\begin{aligned} v(q) &= \int_1^v 1 d\tilde{v} + 1 \\ &= \int_1^v \frac{\partial_v r}{1-\mu} d\tilde{v} + 1 \\ &\geq \tilde{d}^{-1}(r(1,v) - r(1,1)) + 1, \end{aligned}$$

Thus, as $q \to \mathcal{I}^+$, it follows that, since $r(q) \to \infty$, we must have $v(q) \to \infty$. On the other hand,

$$\begin{aligned} u(\mathcal{C}_{out} \cap \mathcal{H}^+) &= \int_1^u d\tilde{u} + 1 \\ &= \int_1^u (-\nu)(u,1) d\tilde{u} + 1 \\ &= r(p) - r(\mathcal{C}_{out} \cap \mathcal{H}^+) + 1 \\ &\equiv U. \end{aligned}$$

Thus, with respect to these coordinates, we have

$$\mathcal{D} = [1, U] \times [1, \infty)$$

and

$$\mathcal{H}^+ = \{U\} \times [1, \infty).$$

4.2 Monotonicity

Define the constants

$$\varpi_1 = \varpi(U, 1)$$

$$r_1 = r(U, 1) \tag{36}$$

and

$$M = \sup_{\mathcal{C}_{out}} \varpi. \tag{37}$$

By Assumptions $\mathbf{A}'$ and $\mathbf{E}'$, it follows that both $r_1 > 0$ and $M > 0$. We have the following

Proposition 4.1. *In $\mathcal{D}$, we have*

$$1 \geq \kappa > 0 \tag{38}$$

$$1 - \mu \geq 0 \tag{39}$$

$$\partial_u \varpi \leq 0, \tag{40}$$

$$\partial_v \varpi \geq 0. \tag{41}$$

$$\partial_u \kappa \leq 0, \tag{42}$$

$$r \geq r_1, \tag{43}$$

$$\varpi_1 \leq \varpi \leq M. \tag{44}$$

In $J^-(\mathcal{I}^+) \cap \mathcal{D}$, we have in addition

$$1 - \mu > 0, \tag{45}$$

$$\frac{\nu}{1-\mu} < 0 \tag{46}$$

$$\partial_v \left(\frac{\nu}{1-\mu} \right) \leq 0. \tag{47}$$

Proof. Inequality (43) is immediate from (14), (13). In view of (28), inequality (14) immediately implies (42) and the left hand side of (38). On any segment $[1, u] \times \{v\}$, where $u \leq U$, $r^{-1}\frac{\zeta^2}{\nu}$ is necessarily uniformly bounded by continuity and compactness. Thus, integrating (28),

$$\kappa(u, v) = \kappa(1, v) e^{\int_1^u r^{-1}\nu^{-1}\zeta^2(u',v)du'} > 0,$$

i.e., we have obtained the right hand side of (38). Together with (13), this yields (39), with (15), this yields (45), and with (14), this yields (46), (47). The remaining inequalities follow immediately from (26), (27), (33). □

We note that all inequalities in the above proposition are in fact independent of our normalization of coordinates (u, v), except for the left hand side of (38), which depends on our normalization of v.

4.3 The energy estimate

In view of (40), (41), we can view equations (26) and (27) as yielding an energy *estimate* for the scalar field: For any null segment $\{u\} \times [v_1, v_2]$, integrating (27), we obtain

$$\begin{aligned} \int_{v_1}^{v_2} \frac{1}{2}\theta^2\kappa^{-1}(u,v)dv &= \varpi(u,v_2) - \varpi(u,v_1) \\ &\leq M - \varpi_1 \end{aligned} \tag{48}$$

and similarly, integrating (27) on any $[u_1, u_2] \times \{v\}$,

$$-\int_{u_1}^{u_2} \frac{1}{2}\zeta^2 \frac{1-\mu}{\nu}(u,v)du \leq M - \varpi_1. \tag{49}$$

In fact, as will be discussed in Section 11, the above estimates are intimately related to the energy estimate arising from Noether's theorem applied to a scalar field on a fixed spherically symmetric *static* background.

Although the energy estimates (48), (49) will certainly play an important role at all levels in this paper, central to our proof will be sharper estimates, taylored to the geometry of various subregions of $\mathcal{D}$.

4.4 An "almost" Riemann invariant

Where r is large, we shall be able to exploit the quantity $\partial_v(r\phi) = \phi\lambda + \theta$. Recall that this quantity is a "Riemann invariant" in the case of the spherically symmetric wave equation on a fixed Minkowski background, i.e. it satisfies $\partial_u(\phi\lambda + \theta) = 0$. From (29), (31), we obtain here

$$\partial_u(\phi\lambda + \theta) = \frac{\phi}{r^2} 2\nu\kappa \left(\varpi - \frac{e^2}{r}\right). \tag{50}$$

While the right hand side of (50) does not vanish, the r^{-2} factor, together with our assumption (19), allows us to show that the quantity $\phi\lambda + \theta$ has better decay in r than either ϕ or θ alone. We shall see already the important role of (50) in the estimates of Section 5. The equation will be further exploited in Section 7.

4.5 The red-shift effect

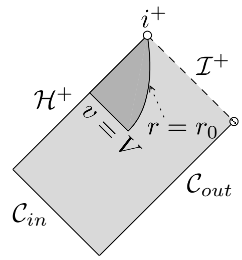
In a neighborhood of $\mathcal{H}^+$, where $r \sim r_+$, very different estimates will apply, estimates which have no analogue in Minkowski space. These estimates are intimately related to the celebrated "red-shift effect", the shift to the lower end of the spectrum for radiation emitted by an observer crossing $\mathcal{H}^+$, as observed by an observer travelling to i^+, or crossing $\mathcal{H}^+$ at a later advanced time. To understand the analytic content of this effect in the context of our system, consider the equation satisfied by the quantity $\frac{\zeta}{\nu}$ appearing in (26):

$$\partial_v\left(\frac{\zeta}{\nu}\right) = -\frac{\theta}{r} - \left(\frac{\zeta}{\nu}\right)\left(\frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)\right). \tag{51}$$

(This equation is easily obtained from (32) and (30).) Let us fix some $r_0 > r_+$, and some sufficiently large V, and restrict attention to the region defined by

$$\mathcal{R} = \{r \leq r_0\} \cap \{v \geq V\} \cap \mathcal{D},$$

depicted as the darker-shaded region in the Penrose diagram below[29]:



We will show in Section 5.2 that

$$\kappa > \hat{c} > 0,$$

and (using Assumption $\mathbf{E}'$!) that

$$\varpi - \frac{e^2}{r} > \tilde{c}' > 0$$

throughout this region, for constants $\hat{c}$, $\tilde{c}'$. Thus, since clearly $r_1 \leq r \leq r_0$ in $\mathcal{R}$, we have

$$2r_0^{-2}M \geq \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right) \geq 2\hat{c}\tilde{c}' r_1^{-2},$$

and consequently, integrating (51),

$$\left|\frac{\zeta}{\nu}(u,v)\right| \leq \int_V^v \frac{|\theta|}{r} e^{-2\hat{c}\tilde{c}' r_1^{-2}(v-v^*)} dv^* + \left|\frac{\zeta}{\nu}(u,V)\right| e^{-2\hat{c}\tilde{c}' r_1^{-2}(v-V)}. \tag{52}$$

In Section 5.2, we shall see that (52) immediately gives uniform bounds for $\left|\frac{\zeta}{\nu}\right|$ in $\mathcal{R}$, since the first term on the right can be bounded by (48) and the second term is clearly bounded by $\left|\frac{\zeta}{\nu}(u,V)\right|$. Later on, however, we will want to exploit the fact that if $v \gg V$, then the second term on the right hand side acquires an exponentially decaying factor. Of course, this information is only useful if the first term also can be shown to be suitably small. In Section 6, we shall integrate (52) in characteristic rectangles $\mathcal{W}$ chosen so that we can indeed make use of the exponential decaying factor in a quantitative way. The significance of this estimate will only become apparent in the context of our induction scheme in Section 9.

[29] Note that $\nu < 0$, $\lambda > 0$ implies that constant-r curves in $J^-(\mathcal{I}^+)$ are timelike!

5 Uniform boundedness and decay for large r

Before turning to decay in u and v, which is the ultimate goal of this paper, we shall begin by showing a series of global bounds yielding decay in r for ϕ, θ, and $\lambda\phi + \theta$.

5.1 $r \geq R$

We consider first the region $r \geq R$, for sufficiently large R. Our tools will be energy estimates, as expressed by (49) and (48), and the additional estimate provided by the "good" right hand side of (50).

Note that Assumption $\mathbf{\Gamma}'$ implies in particular that constant-r curves not crossing $\mathcal{H}^+$ are timelike. It is clear from (44) that for large enough R,

$$1 - \mu(u, v) \geq c \tag{53}$$

for all (u, v) such that $r(u, v) \geq R$, where c is some positive constant, depending on R, such that $c \to 1$ as $R \to \infty$.

Proposition 5.1. *For R sufficiently large, in $\{(u, v) : r(u, v) \geq R\}$, we have the bounds*

$$|\phi| \leq \frac{\hat{C}}{r}, \tag{54}$$

$$|\phi\lambda + \theta| \leq \frac{\hat{C}}{r^{\min\{2,\omega\}}}, \tag{55}$$

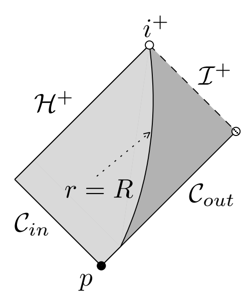
$$\hat{c} \leq \kappa \leq 1, \tag{56}$$

for some constants $\hat{c}, \hat{C} > 0$, where $\hat{c} \to 1$ as $R \to \infty$.

Proof. We will prove the bounds above in several steps. Note first that for

$$R > \max\{r_+, r(p)\}, \tag{57}$$

the set $\{r(u, v) \geq R\}$ has structure depicted by the darker-shaded region of the Penrose diagram below:



In particular, if $(u,v) \in \{r \geq R\}$, then $[1,u] \times \{v\} \subset \{r \geq R\}$. Choose R sufficiently large so that (53) and (57) hold. We estimate, using (49),

$$\begin{aligned}\kappa(u,v) &= \kappa(1,v)e^{\int_1^u \frac{1}{r}\zeta^2 \frac{1-\mu}{\nu}\frac{1}{1-\mu}(u^*,v)du^*} \\ &\geq e^{c^{-1}R^{-1}\int_1^u \zeta^2 \frac{1-\mu}{\nu}du^*} \\ &\geq \hat{c},\end{aligned}$$

where $\hat{c} = \exp\left(2c^{-1}R^{-1}(M - \varpi_1)\right)$. Thus, in view also of (35) and (42), we have established (56).

Note that a bound

$$|\phi(1,v)| \leq \frac{2C}{r}$$

holds on $\mathcal{C}_{out}$ by (18), (19), and the triangle inequality. Integrating the equation

$$\partial_u \phi = \frac{\zeta}{r},$$

from $\mathcal{C}_{out}$ yields

$$\begin{aligned}|\phi(u,v)| &\leq \frac{2C}{r(1,v)} + \sqrt{\int_1^u \zeta^2 \frac{1-\mu}{-\nu}(u^*,v)du^*}\sqrt{\int_1^u \frac{-\nu}{1-\mu}\frac{1}{r^2}(u^*,v)du^*} \\ &\leq \frac{\tilde{C}_1}{\sqrt{r}},\end{aligned}$$

for

$$\tilde{C}_1 = \frac{2C}{\sqrt{r_1}} + \sqrt{2(M-\varpi_1)}\sqrt{r_1^{-1}c^{-1}},$$

where the last inequality follows in view of the bound (53) and the energy estimate (49).

Integrating now (50) from $\mathcal{C}_{out}$, in view of (56), the bound

$$\left|\varpi - \frac{e^2}{r}\right| \leq e^2 r_1^{-1} + M + |\varpi_1|, \tag{58}$$

and (19), we obtain

$$|\phi\lambda + \theta|(u,v) \leq \tilde{C}_2 r^{-\min\{\frac{3}{2},\omega\}},$$

where

$$\tilde{C}_2 = C + \frac{2}{3}\tilde{C}_1(e^2 r_1 + M + |\varpi_1|).$$

By (57), if $r(u,v) \geq R$, then there exists a unique $v_* \leq v$ such that $r(u,v_*) = R$. We can thus integrate the equation

$$\partial_v(r\phi) = \phi\lambda + \theta$$

on $\{u\}\times[v_*,v]$ to obtain

$$\begin{aligned}|r\phi|(u,v) &\leq \frac{\tilde{C}_1 R}{\sqrt{R}} + \left|\int_{v_*}^{v} \tilde{C}_2 r^{-\min\{\frac{3}{2},\omega\}} dv\right| \\ &\leq \tilde{C}_1\sqrt{R} + \max\left\{\frac{2\tilde{C}_2}{\sqrt{R}}, \frac{\tilde{C}_2}{\omega-1}R^{1-\omega}\right\}\end{aligned}$$

and thus

$$|\phi|(u,v) \leq \frac{\tilde{C}_3}{r}. \tag{59}$$

Integrating again (50) using the bound (59) we obtain finally

$$|\lambda\phi+\theta|(u,v) \leq \frac{\tilde{C}_4}{r^{\min\{2,\omega\}}}.$$

Thus (54) and (55) hold with $\hat{C}=\max\{\tilde{C}_3,\tilde{C}_4\}$. $\square$

5.2 $r \leq R$: Uniform boundedness near $\mathcal{H}^+$

We now turn to the region $r \leq R$. We will obtain uniform bounds on $\left|\frac{\zeta}{\nu}\right|$, and then extend our bounds to all of $\mathcal{D}$.

Proposition 5.2. *Let R be as in Proposition 5.1. In the region $\{(u,v) : r(u,v) \leq R\}$, we have a bound*

$$\left|\frac{\zeta}{\nu}\right| \leq \hat{C}, \tag{60}$$

for some $\hat{C} > 0$.

Proof. We note that by Assumptions $\mathbf{B}'$ and $\mathbf{\Gamma}'$ and compactness, it follows that $\sup_{1\leq u\leq U}\left|\frac{\zeta}{\nu}(u,v)\right| \leq C(v)$. Thus, the difficulty involves obtaining uniform estimates in v. We have already seen in Section 4 how we hope to proceed. The main difficulty is to show that the bound (73) actually holds.

We first wish to exclude the case where there exists a v' and a *positive* constant c' such that

$$1-\mu(U,v) \geq c', \tag{61}$$

for all $v \geq v'$.

Recall from $\mathbf{A}'$ the constant r_+, and define

$$\varpi_+ = \sup_{\mathcal{H}^+}\varpi.$$

Note that by (44), $\varpi_+ \leq M$. Assuming for the time being (61), we can choose sufficiently small intervals $[r',r'']$, $[\varpi',\varpi'']$, containing in their interior r_+, ϖ_+,

respectively, so that for $r' \le r^* \le r''$, $\varpi' \le \varpi^* \le \varpi''$, we have $1 - \frac{2\varpi^*}{r^*} + \frac{e^2}{(r^*)^2} \ge \frac{c'}{2}$.

Choose v' such that in addition to (61), we have $r(U, v') \ge r'$, and $\varpi(U, v') \ge \varpi'$, and consider the region

$$\mathcal{R} = \{v \ge v'\} \cap \{\varpi' \le \varpi(u, v) \le \varpi''\} \cap \{r' \le r(u, v) \le r''\}.$$

It is clear from inequalities (14), (13), (40), and (41) that if $(u, v) \in \mathcal{R}$, then

$$\{u\} \times [v', v] \subset \mathcal{R}, [u, U] \times \{v\} \subset \mathcal{R};$$

moreover, the inequality

$$1 - \mu \ge \frac{c'}{2} \tag{62}$$

holds in $\mathcal{R}$. Integrating (51) from $v = v'$, we obtain:

$$\begin{aligned} \left|\frac{\zeta}{\nu}(u, v)\right| \quad &\le \quad \left(\sqrt{\int_{v'}^{v} \frac{1-\mu}{\lambda} \theta^2(u, v^*) dv^*} \sqrt{\int_{v'}^{v} \frac{\lambda}{1-\mu} \frac{1}{r^2}(u, v^*) dv^*}\right. \\ &\qquad \left. + \left|\frac{\zeta}{\nu}(u, v')\right|\right) e^{\int_{v'}^{v} \frac{\lambda}{1-\mu} \frac{2}{r^2}\left|\frac{e^2}{r} - \varpi\right|(u, v^*) dv^*} \\ &\le \quad \left(\sqrt{2(M - \varpi_1)} \sqrt{\int_{v'}^{v} \frac{\lambda}{1-\mu} \frac{1}{r^2}(u, v^*) dv^*} + \left|\frac{\zeta}{\nu}(u, v')\right|\right) \\ &\qquad \cdot e^{\int_{v'}^{v} \frac{\lambda}{1-\mu} \frac{2}{r^2}\left|\frac{e^2}{r} - \varpi\right|(u, v^*) dv^*}. \end{aligned}$$

For $(u, v) \in \mathcal{R}$, it follows from (62) and the bounds on ϖ and r that

$$\left|\frac{\zeta}{\nu}\right| \le \tilde{C}, \tag{63}$$

where

$$\begin{aligned} \tilde{C} \quad &= \quad \left(\sqrt{4c'^{-1}(M - \varpi_1)(r'^{-1} - r''^{-1})} + \sup_u \left|\frac{\zeta}{\nu}(u, v')\right|\right) \cdot \\ &\qquad \cdot \exp\left(2c^{-1}(r'^{-1} - r''^{-1})(e^2 r'^{-1} + M + |\varpi_1|)\right). \end{aligned}$$

Let $\partial\mathcal{R}$ denote the boundary of $\mathcal{R}$ in the topology of $\mathcal{D} \cap \{v > v'\}$. It is clear from the properties of $\mathcal{R}$ discussed above that $\partial\mathcal{R}$ must be a non-empty connected causal curve terminating at i^+. Moreover, we can choose a sequence of points

$$p_i = (u_i, v_i) \in \partial\mathcal{R}$$

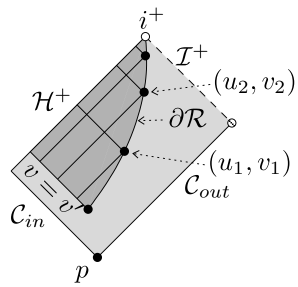
such that $(u_i, v_i) \to (\infty, \infty)$, and either

$$r(u_i, v_i) = r'' \tag{64}$$

or

$$\varpi(u_i, v_i) = \varpi''. \tag{65}$$

Assume first (64). We will show this leads to a contradiction. Refer to the Penrose diagram below:



Equation (64) immediately yields

$$\int_{u_i}^{U} \nu(u, v_i)du \to r_+ - r'', \tag{66}$$

as $i \to \infty$. Since $\{u_i\} \times [v', v_i] \subset \mathcal{R}$, we have the bound

$$\int_{v'}^{v_i} \frac{\lambda}{1-\mu}\frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right)(u_i, v^*)dv^* < 2c^{-1}(r'^{-1} - r''^{-1})(e^2 r'^{-1} + M + |\varpi_1|).$$

Integrating (32) in v, and then integrating in u, we obtain that

$$\begin{aligned}\int_{u_i}^{U} (-\nu)(u, v_i)du \quad &\leq \quad \exp\left(2c^{-1}(r'^{-1} - r''^{-1})(e^2 r'^{-1} + M + |\varpi_1|)\right) \cdot \\ &\quad \cdot \int_{u_i}^{U} (-\nu)(u, v')du \end{aligned} \tag{67}$$

As $i \to \infty$, the domain of integration $[u_i, U] \times \{v'\}$ of the integral on the left shrinks to the point (U, v'), and thus the right hand side of (67) tends to 0. But this contradicts (66). Thus, (64) is impossible.

We must have then (65). But this gives

$$\begin{aligned}\varpi'' - \varpi_+ \quad &\leq \quad \varpi(u_i, v_i) - \varpi(\infty, v_i) \\ &= \quad \int_{u_i}^{U} \frac{1}{2}\left(\frac{\zeta}{\nu}\right)^2 (-\nu)(1-\mu)(u, v_i)du \\ &\leq \quad \frac{1}{2}\tilde{C}^2(1 + 2|\varpi_1|r'^{-1} + e^2 r'^{-2}) \int_{u_i}^{U} (-\nu)(u, v_i)du. \end{aligned}$$

Since, by (67), the right hand side of the above tends to 0, whereas the left hand side is a strictly positive number independent of i, we immediately obtain a contradiction.

We have thus excluded (61) and can assume in what follows that there is a sequence of points $(U, \hat{v}_i) \in \mathcal{H}^+$ such that $1 - \mu(U, \hat{v}_i) \to 0$. But then by (41),

it follows that in fact $1-\mu(U,v)\to 0$. In particular:

$$\varpi_+ = \frac{r_+}{2} + \frac{e^2}{2r_+}.$$

This implies that $|e| \leq \varpi_+$, and either

$$r_+ = \varpi_+ + \sqrt{\varpi_+^2 - e^2} \tag{68}$$

or

$$r_+ = \varpi_+ - \sqrt{\varpi_+^2 - e^2}. \tag{69}$$

The possibilities (68) and (69) coincide if and only if $e = \varpi_+$. This would give, however, $r_+ = |e|$, which contradicts Assumption $\mathbf{\Sigma T'}$. Thus (68) and (69) are necessarily distinct. Note that if $e = 0$ we certainly have (68). We show now that in general, (69) leads to a contradiction.[30]

In the case (69), we have by a simple computation that

$$\varpi_+ - \frac{e^2}{r_+} < 0,$$

and thus, it is clear that we can choose $0 < r_2 < r_+$, $0 < \varpi_2 < \varpi_+$ close enough to r_+, ϖ_+, respectively, so that

$$\varpi^* - \frac{e^2}{r^*} < -2\tilde{c},$$

for some constant $\tilde{c}$ and for all $r^* \in [r_2, r_+]$, $\varpi^* \in [\varpi_2, \varpi_+]$, and we can then choose v' sufficiently large so that $r(U,v') \geq r_2$, $\varpi(U,v') \geq \varpi_2$.

Fix an $\epsilon > 0$. By the inequalities $1-\mu(u,v) \geq 0$ and $r(u,v) \geq r_2$, for $v \geq v'$, it follows that for $(u,v) \in \{r(u,v) \leq r_+ + \epsilon\} \cap \{v \geq v'\}$, we have $\varpi(u,v) \leq \varpi_+ + \tilde{\epsilon}$, where

$$\tilde{\epsilon} = \epsilon\left(\varpi_+ + \frac{2e^2(r_+ + \epsilon)^2\epsilon}{r_+^2 r_2^2}\right).$$

If ϵ is chosen sufficiently small, we thus have

$$\varpi - \frac{e^2}{r} < -\tilde{c}, \tag{70}$$

in $\{r \leq r_+ + \epsilon\} \cap \{v \geq v'\}$.

The timelike curve $r = r_+ + \epsilon$ cannot cross $\mathcal{H}^+$ so must "terminate" at i^+. Choose a sequence of points $(u_i, v_i) \to i^+$ on $r = r_+ + \epsilon$, with $v_i \geq v'$. Clearly, we must have

$$\int_{u_i}^{U} (-\nu)(u, v_i)du \to \epsilon. \tag{71}$$

[30] If we modify our initial assumptions by requiring in addition $m \geq 0$ on $\mathcal{C}_{in}$, then (68) follows immediately from the inequality $\partial_v m \geq 0$ in $\mathcal{D}$.

On the other hand, integrating equation (32), noting that its right hand side is positive in view of (70), we obtain that

$$\int_{u_i}^{U} (-\nu)(u, v_i)du \leq \int_{u_i}^{U} (-\nu)(u, v')du. \tag{72}$$

Since the right hand side of (72) tends to 0, (72) contradicts (71). Thus (69) is impossible.

We are thus in the case (68). Again, choosing $r_3 < r_+$, $0 < \varpi_3 < \varpi_+$ sufficiently close to r_+, ϖ_+, respectively, we can arrange so that

$$\varpi_3 - \frac{e^2}{r_3} = \tilde{c}' > 0,$$

and $r(U, v') \geq r_3$, $\varpi(U, v') \geq \varpi_3$, for some v'. In $[1, U] \times [v', \infty)$, it follows from (14), (13), (40), and (41) that

$$\varpi - \frac{e^2}{r} \geq \varpi_3 - \frac{e^2}{r_3} = \tilde{c}'. \tag{73}$$

Thus, for $v \geq v'$, we have, integrating (51), that

$$\begin{aligned}
\left|\frac{\zeta}{\nu}(u,v)\right| \quad &\leq \quad \left|\int_{v'}^{v} \frac{\theta}{r}(u, v^*) e^{\int_{v^*}^{v} -\frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)(u,\tilde{v})d\tilde{v}} dv^*\right| \\
&\qquad + \left|\frac{\zeta}{\nu}\right|(u, v') e^{\int_{v'}^{v} -\frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)(u,v^*)dv^*} \\
&\leq \quad r_1^{-1}\sqrt{\int_{v'}^{v} \kappa^{-1}\theta^2(u, v^*)dv^*}\sqrt{\int_{v'}^{v} \kappa(u, v^*) e^{2\int_{v^*}^{v} -\frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)(u,\tilde{v})d\tilde{v}}} \\
&\qquad + \left|\frac{\zeta}{\nu}\right|(u, v') e^{\int_{v'}^{v} \kappa\frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right)(u,v^*)dv^*} \\
&\leq \quad r_1^{-1}\sqrt{2(M - \varpi_3)}\sqrt{\int_{v'}^{v} \kappa(u, v^*) e^{-4\tilde{c}' R^{-2}\int_{v^*}^{v} \kappa(u,\tilde{v})d\tilde{v}}} \\
&\qquad + \sup_{1\leq u\leq U} \left|\frac{\zeta}{\nu}\right|(u, v').
\end{aligned}$$

Setting

$$\hat{C} = \sqrt{2(M - \varpi_3)}\left(2r_1 R\sqrt{\tilde{c}}\right)^{-1} + \sup_{1\leq u\leq U, 1\leq v\leq v'} \left|\frac{\zeta}{\nu}\right|(u, v),$$

we obtain (60) throught $\{r \leq R\}$. This completes the proof. $\Box$

Finally, we have the following

Proposition 5.3. *The inequalities (54), (55), (56), and (60) hold throughout $\mathcal{D}$, for some constants $\hat{C}, \hat{c} > 0$.*

Proof. The bounds (54), (55), and (56) applied to the region $\{r(u,v) \leq R\}$ follow immediately by integrating the equations:

$$\partial_u \phi = \frac{\zeta}{\nu}\frac{\nu}{r},$$

$$\partial_u \theta = -\frac{\zeta}{\nu}\frac{\nu}{r}\kappa(1-\mu),$$

and (28) to the future from the curve $r = R$, in view of the previous two propositions and the bounds $r \geq r_1$, $|1-\mu| \leq 1 + 2|\varpi_1|r_1^{-1} + e^2 r_1^{-2}$. The bound (60) applied to $r \geq R$ follows by integrating (51) to the right from $r = R$, in view of the bounds (58), (53). $\square$

5.3 A new retarded time u

Before continuing, we will use the results of this section to renormalize our retarded time coordinate u. This renormalization, although not essential, will be useful, because it emphasizes a certain symmetry between the u and the v directions which will play an important role later on. It has the disadvantage, however, that it will put $\mathcal{H}^+$ "at" $u = \infty$. As we shall see in this section, we can easily get around this disadvantage by employing a simple convention.

Choose R as in Proposition 5.1. In the region $\{r(u,v) \geq R\}$, by (53) and the bound (58), we have a uniform bound

$$\left| \int_v^{v'} \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)(u,v^*)dv^* \right| \leq 2R^{-1}c^{-1}(e^2 r_1^{-1} + M + |\varpi_1|), \tag{74}$$

for any $v' > v$. Thus, defining, for $u < U$, the function $\tilde{f}(u)$ by

$$\begin{aligned}\tilde{f}(u) &= \int_1^{\infty} \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)(u,v^*)dv^* \\ &= \int_1^{\hat{v}(u)} \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)(u,v^*)dv^* \\ &\quad + \int_{\hat{v}}^{\infty} \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)(u,v^*)dv^*,\end{aligned}$$

where $\hat{v}(u)$ is defined by $r(u,\hat{v}(u)) = R$, it is clear that $\tilde{f}(u)$ is finite (on account of (74) and the fact that the domain of integration in the first term on the right is finite, and so is the integrand) and (by an easy argument) continuously differentiable. Define now a new null coordinate u^* by

$$u^*(u) = \int_1^u e^{-\tilde{f}(u)}du + 1.$$

This clearly defines a regular change of coordinates for $u < U$. Moreover $\nu^* = \partial_{u^*} r \to -1$ as $v \to \infty$, and this defines a continuous (in the topology of the

Penrose diagram) extension of ν^* to $\mathcal{I}^+$. We will say that u^* is retarded time normalized at null infinity $\mathcal{I}^+$.

In what follows, we shall drop the $*$ and refer to our new coordinate as u, and ν^*, etc., as ν. Note that the bound (74) implies for this new coordinate that in the region $r \geq R$ we have

$$|\nu| \leq \exp\left(2R^{-1}c^{-1}(e^2 r_1^{-1} + M + |\varpi_1|)\right). \tag{75}$$

Proposition 5.4. *The new u-coordinate covers $J^-(\mathcal{I}^+) \cap \mathcal{D}$ with range $[1, \infty)$.*

Proof. Integrating (11) with (75) we have

$$r(u,v) \geq r(1,v) - \exp\left(2R^{-1}c^{-1}(e^2 r_1^{-1} + M + |\varpi_1|)\right) u,$$

for (u,v) in the region $r \geq R$. By Assumptions $\mathbf{A}'$ and $\mathbf{\Gamma}'$, for $R > r_+$, $r(u,v) = R$ is a timelike curve which terminates at i^+, in the topology of the Penrose diagram. In particular, there exist points on this curve with arbitrarily large v. But since $r(1,v) \to \infty$ as $v \to \infty$, by Assumption $\mathbf{A}'$, it follows that for these points $u \to \infty$. $\Box$.

An elaboration of the method of the proof of the above proposition immediately yields that null infinity is *complete* in the sense of Christodoulou [16]. See [26].

As remarked above, the disadvantage of this new coordinate system is that the event horizon $\mathcal{H}^+$ is not covered. We can formally parametrize it, however, by (∞, v). Moreover, since $\frac{\zeta}{\nu}$, λ, θ do not depend on the choice of u-coordinate, one can make sense of expressions like $\frac{\zeta}{\nu}(\infty, 0)$, $\int_0^\infty \lambda(\infty, v)dv$. These can be formally defined by changing back to the old u-coordinate, but we shall save ourselves this inconvenience.

With respect to our new coordinates, we have the following bounds:

Proposition 5.5. *Fix $r_0 > r_+$. There exist constants c and G, depending on r_0, such that the inequalities*

$$c \leq 1 - \mu \leq 1, \tag{76}$$

$$0 < G \leq \lambda \leq 1, \tag{77}$$

$$-1 \leq \nu \leq -G \tag{78}$$

hold in the region $\{r \geq r_0\} \cap \{v \geq v'\}$, where v' is as defined before (73). As $r_0 \to \infty$ then G and c can be chosen arbitrarily close to 1.

Proof. The right inequality of (76) arises from

$$\begin{aligned} 1 - \mu &= 1 - \frac{2m}{r} \\ &= 1 - \frac{e^2}{r^2} + \frac{2}{r}\left(\frac{e^2}{r} - \varpi\right) \\ &\leq 1 - \frac{e^2}{r^2} - \frac{2\tilde{c}'}{r} \\ &< 1, \end{aligned}$$

where for the third inequality we use (73).

To show the left inequality of (76), as well as (77) and (78), we fix an $\epsilon > 0$, and choose $r_+ < r'' < r_0$ sufficiently close to r_+ so that

$$1 - \frac{2\varpi_*}{r} + \frac{e^2}{r_*} \leq \frac{\epsilon}{2\hat{C}^2}, \tag{79}$$

$$r_* - \left(\varpi_* - \sqrt{\varpi_*^2 - e^2}\right) > \epsilon \tag{80}$$

for any

$$r_* \in [r_+ - (r'' - r_+), r''], \varpi_* \in [\varpi_+ - (r'' - r_+)4^{-1}\hat{C}^2, \varpi_+ - (r'' - r_+)4^{-1}\hat{C}^2].$$

Let $\tilde{v}$ be such that $r(\infty, \tilde{v}) \geq r_+ - (r'' - r_+)$, $\varpi(\infty, \tilde{v}) \geq \varpi_+ - (r'' - r_+)4^{-1}\hat{C}^2$, and consider the region

$$\mathcal{R} = \{v \geq \tilde{v}\} \cap \{r(u, v) \leq r''\}.$$

Integrating (26) from (∞, v), in view of the right inequality of (76), and Proposition 5.3, it follows that for $(u, v) \in \mathcal{R}$,

$$\varpi_+ - (r'' - r_+)4^{-1}\hat{C}^2 \leq \varpi(u, v) \leq \varpi_+ - (r'' - r_+)4^{-1}\hat{C}^2$$

and thus (79) holds. Integrating (26) again, but using (79) we obtain

$$|\varpi(u, v) - \varpi_+| \leq \frac{\epsilon}{2}(r'' - r_+). \tag{81}$$

Consider the quantity $\varpi + \sqrt{\varpi^2 - e^2} - r$ at (u, v) on the curve $r = r''$. We have

$$\begin{aligned} \varpi + \sqrt{\varpi^2 - e^2} - r &= \varpi - \varpi_+ + \sqrt{\varpi^2 - e^2} - \sqrt{\varpi_+^2 - e^2} - r + r_+ \\ &\leq K|\varpi - \varpi_+| - (r'' - r_+) \qquad (82) \\ &\leq \left(\frac{K\epsilon}{2} - 1\right)(r'' - r_+), \qquad (83) \end{aligned}$$

for a positive constant K easily computed, where for the last inequality we have used (81). Thus for $r(u, v) = r''$, we have

$$\begin{aligned} 1 - \mu(u, v) &= \frac{1}{r^2}\left(r - \left(\varpi + \sqrt{\varpi^2 - e^2}\right)\right)\left(r - \left(\varpi - \sqrt{\varpi^2 - e^2}\right)\right) \\ &\geq r''^{-2}\left(1 - \frac{K\epsilon}{2}\right)(r'' - r_+)\epsilon \end{aligned}$$

where for the last inequality we have used (80) and (82). Choosing $\epsilon > 0$ sufficiently small, the above gives a bound

$$1 - \mu(u, v) \geq \epsilon' > 0 \tag{84}$$

for all (u, v) on $r = r''$.

From the bound (84) together with the lower bound on κ from Proposition 5.3, we obtain, on the curve $r = r''$, a bound

$$\lambda = \kappa(1-\mu) \geq \hat{c}\epsilon'.$$

But from the equation (31) and (73), it follows that $\partial_u \lambda \leq 0$. Thus, we have (77) in $r \geq r''$ as long as $G \leq \hat{c}\epsilon'$. From this, we immediately derive

$$1 - \mu = \kappa^{-1}\lambda \geq \hat{c}\epsilon'$$

for $r \geq r''$, which yields the left inequality of (76). Finally, (78) follows after integration of (32) from $v = \infty$, in view of (76).

For r_0 sufficiently large, we have the alternative bound

$$1 - \mu \geq \lambda = \kappa(1-\mu) \geq \hat{c}c,$$

where c is the constant of (53), and thus

$$-\nu \geq \exp(2r_0^{-1}\hat{c}M).$$

It follows that we can take $G \to 1$ as $r_0 \to \infty$. $\Box$

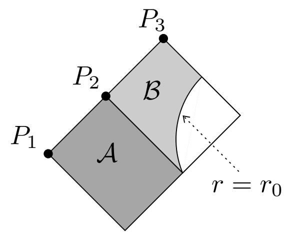
6 A red-shift estimate on long rectangles

We have seen in Section 5.2 the importance of the "red-shift" effect in obtaining bounds for $\frac{\zeta}{\nu}$. In this section, we seek to take advantage of the exponential factor in the second term of (52). For this, we shall restrict our estimates to a characteristic rectangle $\mathcal{W}$, and seek to formulate assumptions on the boundary of $\mathcal{W}$ that optimize the bounds we can obtain for the two terms on the right hand side of (52), and thus, upon integration, for $\frac{\zeta}{\nu}$. These assumptions will involve, on the one hand, the v-dimensions of the rectangle (see (85)), which will allow us to control the second term of (52) using the exponential factor, and on the other hand, an *a priori* bound on the mass difference (see (86)) through one of the outgoing null edges of $\partial\mathcal{W}$, which will allow us to control the first term through an energy estimate argument. Both assumptions are formulated with respect to a constant D, and the final bounds in $\mathcal{W}$ depend on this constant and the value of r. In particular, the power law dependence on r in the bound (95) is fundamental.

The rectangle-estimate of this section will be used in the induction argument of Section 9. The reader impatient to understand the structure of the argument can turn there now.

We introduce the notation $\mathcal{D}' = \mathcal{D} \cap \{v \geq v'\}$, where v' is chosen so that (73) holds on $\mathcal{D}'$. When we say in what follows that a constant C_3 depends "only" on given constants C_1 and C_2, this is meant *in addition* to the constants C of bounds (18) and (19), $\hat{C}$ and $\hat{c}$ of Proposition 5.3, $\tilde{c}'$ of (73), and r_3, M, and $\varpi_3 > 0$.

Proposition 6.1. *Let r_0 be a constant $r_0 > r_1$, where r_1 is the constant of (36), and let D be any constant $D > 1$. Consider a characteristic rectangle $\mathcal{W} \subset \mathcal{D}'$ with Penrose diagram depicted below*[31]*:*



Assume

$$v(P_2) - v(P_1) \geq \frac{r_0^2}{4\tilde{c}'\hat{c}} \log D, \tag{85}$$

$$\varpi(P_3) - \varpi(P_1) \leq D^{-1}, \tag{86}$$

and

$$\triangle_{\mathcal{A}\cup\mathcal{B}} r \leq \epsilon \tag{87}$$

for some $\epsilon > 0$ satisfying

$$\sqrt{\epsilon} < r_0^{-1} r_1 \sqrt{\tilde{c}'\hat{c}}, \tag{88}$$

and where for $\mathcal{C} \subset \mathcal{D}$, we define

$$\triangle_{\mathcal{C}} f = \sup_{p_i \in \mathcal{C}} |f(p_1) - f(p_2)|.$$

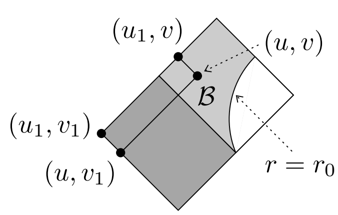
Then, there exists a positive constant $\tilde{C}$ depending only on r_0 and ϵ such that

$$\left| \frac{\zeta}{\nu} \right| \leq \tilde{C} D^{-\frac{1}{2}} \tag{89}$$

in $\mathcal{B}$, and

$$\triangle_{\mathcal{B}} \varpi \leq \tilde{C} D^{-1}. \tag{90}$$

Proof. Consider $(u, v) \in \mathcal{B}$. We will derive a pointwise bound for $\left|\frac{\zeta}{\nu}(u, v)\right|$. Let P_1 be given by (u_1, v_1):



[31] Translation: Let $P_1 = (u_1, v_1)$, $P_2 = (u_1, v_2)$, $P_3 = (u_1, v_3)$, $v_3 > v_2 > v_1$, let u_2 be such that $r(u_2, v_2) = r_0$, define $\mathcal{W} = [u_2, u_1] \times [v_1, v_3]$, $\mathcal{A} = [u_2, u_1] \times [v_1, v_2]$, $\mathcal{B} = [u_2, u_1] \times [v_2, v_3] \cap \{r \leq r_0\}$. Note that we allow $u_1 = \infty$.

Integrating the equation (51) we obtain

$$\frac{\zeta}{\nu}(u,v)=\int_{v_1}^{v}-\frac{\theta}{r}(u,\tilde v)e^{\int_{\tilde v}^{v}-\frac{2\kappa}{r^2}\left(\varpi-\frac{e^2}{r}\right)}d\tilde v+\frac{\zeta}{\nu}(u,v_1)e^{\int_{v_1}^{v}-\frac{2\kappa}{r^2}\left(\varpi-\frac{e^2}{r}\right)(u,\tilde v)d\tilde v}.$$

Thus,

$$\begin{aligned}
\left|\frac{\zeta}{\nu}\right|(u,v) &\le r_1^{-1}\sqrt{\int_{v_1}^{v}\theta^2(u,\tilde v)d\tilde v}\sqrt{\int_{v_1}^{v}e^{2\int_{\tilde v}^{v}-\frac{2\kappa}{r^2}\left(\varpi-\frac{e^2}{r}\right)}}+\hat C e^{-2\tilde c'\hat c r_0^{-2}(v-v_1)}\\
&\le r_1^{-1}\sqrt{\int_{v_1}^{v}\theta^2(u,\tilde v)d\tilde v}\sqrt{\int_{v_1}^{v}e^{-4\tilde c'\hat c r_0^{-2}(v-\tilde v)}d\tilde v}+\hat C e^{-\frac{1}{2}\log D}\\
&\le r_1^{-1}\sqrt{\int_{v_1}^{v}\theta^2(u,\tilde v)d\tilde v}\sqrt{(4\tilde c'\hat c r_0^{-2})^{-1}\left(1-e^{-\sqrt{D}}\right)}\\
&\quad+\hat C e^{-\log\sqrt{D}}\\
&\le r_0\left(2r_1\sqrt{\tilde c'\hat c}\right)^{-1}\sqrt{\int_{v_1}^{v}\theta^2 d\tilde v}+\hat C e^{-\sqrt{D}}.
\end{aligned}$$

Here we have used (85), (73), and the bound $r\le r_0$ on $\mathcal{B}$, manifest from the Penrose diagram. On the other hand, by (86) and (38), we have that

$$\begin{aligned}
\frac{1}{2}\int_{v_1}^{v}\theta^2(u,\tilde v)d\tilde v &\le \varpi(u,v)-\varpi(u,v_1)\\
&\le (\varpi(u,v)-\varpi(u_1,v))+(\varpi(u_1,v)-\varpi(u_1,v_1))\\
&\le \int_{u_1}^{u}\left(\frac{\zeta}{\nu}\right)^2(1-\mu)(-\nu)(\tilde u,v)d\tilde u+D^{-1}\\
&\le \epsilon\left(\sup_{u_*\in[u,u_1]}\left(\frac{\zeta}{\nu}\right)^2(u_*,v)\right)+D^{-1}.
\end{aligned}$$

Putting these two estimates together yields

$$\begin{aligned}
\left|\frac{\zeta}{\nu}\right|(u,v) &\le r_0\left(2r_1\sqrt{\tilde c'\hat c}\right)^{-1}\sqrt{\epsilon\sup_{u_*\in[u,u_1]}\left(\frac{\zeta}{\nu}\right)^2(u_*,v)+D^{-1}}\\
&\quad+\hat C e^{-\sqrt{D}}\\
&\le r_0\left(2r_1\sqrt{\tilde c'\hat c}\right)^{-1}\left(\sqrt{\epsilon}\sup_{u_*\in[u,u_1]}\left|\frac{\zeta}{\nu}\right|(u_*,v)+D^{-\frac{1}{2}}\right)\\
&\quad+\hat C e^{-\sqrt{D}}\\
&\le \tilde\epsilon\sup_{u_*\in[u,u_1]}\left|\frac{\zeta}{\nu}\right|(u_*,v)+C'D^{-\frac{1}{2}},
\end{aligned}$$

where

$$\tilde\epsilon=\frac{\sqrt{\epsilon}r_0}{2r_1\sqrt{\tilde c'\hat c}}. \tag{91}$$

Now clearly, the above estimate also applies to $\sup_{u_*\in[u,u_1]}\left|\frac{\zeta}{\nu}\right|(u_*,v)$, i.e., we have in addition

$$\sup_{u_*\in[u,u_1]}\left|\frac{\zeta}{\nu}\right|(u_*,v)\leq\tilde{\epsilon}\sup_{u_*\in[u,u_1]}\left|\frac{\zeta}{\nu}\right|(u_*,v)+C'D^{-\frac{1}{2}},$$

and thus, in view of (88),

$$\sup_{u_*\in[u,u_1]}\left|\frac{\zeta}{\nu}\right|(u_*,v)\leq\frac{C'D^{-\frac{1}{2}}}{1-\tilde{\epsilon}}. \tag{92}$$

This gives in particular (89). The bound (90) follows immediately by integrating (26) using (92), together with the bound (86). □

In the proof of the next proposition, fix $r_0>r_+$ and recall the constant G_{r_0}, defined in Proposition 5.5.

Proposition 6.2. *Let $R\geq r_0>r_+$, consider a characteristic rectangle $\mathcal{W}\subset\mathcal{D}$ and suppose that on $\mathcal{W}\cap\{r=R\}$ we have the bound*

$$|\zeta|\leq\bar{A},$$

as well as the bound

$$\triangle_{\mathcal{W}\cap\{r\leq R\}}\varpi\leq\bar{A}^2R,$$

for some positive constant $\bar{A}$. Then, there is a constant $\tilde{\beta}>1$, depending only on r_0, such that in

$$\{R\leq r\leq\tilde{\beta}R\}\cap\mathcal{W}$$

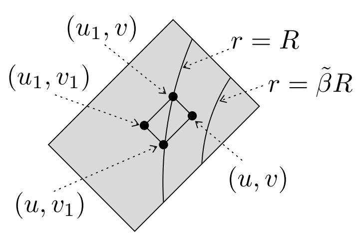
we have the bound

$$|\zeta|\leq 3\bar{A},$$

and

$$\triangle_{\mathcal{W}\cap\{r\leq\tilde{\beta}R\}}\varpi\leq 9\bar{A}^2\tilde{\beta}R.$$

Proof. Let (u,v) be any point in $\mathcal{W}\cap\{R\leq r\leq\tilde{\beta}R\}$, and let (u_1,v_1) be as in the Penrose diagram below:



Integrating the equation

$$\partial_v\zeta=-\frac{\theta}{r}\nu$$

on $\{u\}\times[v_1,v]$ we obtain

$$\begin{aligned}
|\zeta|(u,v) &\leq \int_{v_1}^{v}\left|\frac{\theta\nu}{r}(u,\tilde v)\right|d\tilde v+|\zeta(u,v_1)|\\
&\leq \sqrt{G^{-1}}\sqrt{\int_{v_1}^{v}\theta^2(u,\tilde v)d\tilde v}\sqrt{\int_{v_1}^{v}\frac{\lambda}{r^2}(u,\tilde v)d\tilde v}+\bar A\\
&\leq \sqrt{G^{-1}}\sqrt{\int_{v_1}^{v}\theta^2(u,\tilde v)d\tilde v}\sqrt{\frac{\tilde\beta-1}{\tilde\beta R}}+\bar A.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{1}{2}\int_{v_1}^{v}\theta^2(u,\tilde v)d\tilde v &\leq \varpi(u,v)-\varpi(u_1,v_1)\\
&= (\varpi(u,v)-\varpi(u_1,v))+(\varpi(u_1,v)-\varpi(u_1,v_1))\\
&\leq \int_{u}^{u_1}\left(\frac{\zeta}{\nu}\right)^2(1-\mu)(-\nu)(\tilde u,v)d\tilde u+R\bar A^2\\
&\leq \left(\sup_{u_*\in[u,u_1]}\left(\frac{\zeta}{\nu}\right)^2(u_*,v)\right)\int_{u}^{u_1}|\nu|(\tilde u,v)d\tilde u+R\bar A^2\\
&\leq (\tilde\beta-1)R\left(\sup_{u_*\in[u,u_1]}\left|\frac{\zeta}{\nu}\right|(u_*,v)\right)^2+R\bar A^2.
\end{aligned}$$

Combining this with the previous, we obtain

$$\begin{aligned}
|\zeta|(u,v) &\leq \sqrt{G^{-1}}\sqrt{2(\tilde\beta-1)R\left(\sup_{u_*\in[u,u_1]}\left|\frac{\zeta}{\nu}\right|(u_*,v)\right)^2+2R\bar C^2}\sqrt{\frac{\tilde\beta-1}{\tilde\beta R}}+\bar A\\
&\leq \frac{1}{2}\sup_{u_*\in[u,u_1]}|\zeta|(u_*,v)+\frac{\bar A}{2}+\bar A, \qquad (93)
\end{aligned}$$

where to obtain the first two terms on the right in the latter inequality, we have chosen $\tilde\beta>1$ such that

$$\tilde\beta-1\leq\frac{G}{8}. \qquad (94)$$

Since the estimate (93) is clearly also valid when one replaces $|\zeta|$ with

$$\sup_{u_*\in[u,u_1]}|\zeta|,$$

we obtain:

$$\sup_{u_*\in[u,u_1]}|\zeta|(u_*,v)\leq\frac{1}{2}\sup_{u_*\in[u,u_1]}|\zeta|(u_*,v)+\frac{\bar A}{2}+\bar A,$$

and thus

$$\sup_{u_*\in[u,u_1]}|\zeta|(u_*,v)\leq 3\bar A.$$

Finally, we have by the above

$$\begin{aligned}\frac{1}{2}\int_u^{u_1}\left(\frac{\zeta}{\nu}\right)^2(-\nu)(1-\mu)(\tilde{u},v)d\tilde{u} &\leq \frac{1}{2}G_1^{-2}9\bar{A}^2R(\tilde{\beta}-1)\\ &\leq 8\tilde{\beta}\bar{A}^2R,\end{aligned}$$

where the second estimate follows provided that $\tilde{\beta}$ is chosen to satisfy, in addition to (94), the inequality

$$\tilde{\beta}-1\leq\frac{16G^2}{9}.$$

Thus we have

$$\begin{aligned}\triangle_{\mathcal{W}\cap\{r\leq\tilde{\beta}R\}}\varpi &\leq \triangle_{\mathcal{W}\cap\{r\leq R\}}\varpi+8\tilde{\beta}\bar{A}^2R\\ &\leq \tilde{C}^2R+8\tilde{\beta}\bar{A}^2R\\ &\leq 9\bar{A}^2\tilde{\beta}R,\end{aligned}$$

and this completes the proof of the Proposition. $\Box$

Proposition 6.3. *Let $r_0>r_+$, and assume we have a characteristic rectangle $\mathcal{W}$ satisfying the assumptions of Proposition 6.1. Then, it follows that we have the estimate*

$$\left|\frac{\zeta}{\nu}\right|\leq Hr^\alpha D^{-\frac{1}{2}} \tag{95}$$

in $\mathcal{W}\setminus\mathcal{A}$, for some constant $H>0$ depending only on r_0 and ϵ, where

$$\alpha=\frac{\log 3}{\log\tilde{\beta}}.$$

and $\tilde{\beta}$ is as in Proposition 6.2.

Proof. From the result of Proposition 6.1, the assumptions of Proposition 6.2 hold on $r=r_0$, where

$$\bar{A}=\max\left\{\tilde{C}D^{-\frac{1}{2}},\sqrt{\tilde{C}r_0^{-1}}D^{-\frac{1}{2}}\right\}.$$

Denote $r_k=\tilde{\beta}^kr_0$. By iterating the result of Proposition 6.2, one obtains for points with $r_{k-1}\leq r(u,v)\leq r_k$, a bound

$$|\zeta(u,v)|\leq 3^k\max\left\{\tilde{C},\sqrt{\tilde{C}r_0^{-1}}\right\}D^{-\frac{1}{2}}.$$

From

$$k=\frac{\log r_k-\log r_0}{\log\tilde{\beta}},$$

we have

$$\begin{aligned}\left|\frac{\zeta}{\nu}(u,v)\right| &\leq G^{-1}3^{-\log r_0(\log\tilde{\beta})^{-1}}\max\left\{\tilde{C},\sqrt{\tilde{C}r_0^{-1}}\right\}3^{(\log\tilde{\beta})^{-1}\log r_k}D^{-\frac{1}{2}}\\ &\leq G^{-1}3^{-\log r_0(\log\tilde{\beta})^{-1}}\max\left\{\tilde{C},\sqrt{\tilde{C}r_0^{-1}}\right\}e^{\log r_k^{(\log 3)(\log\tilde{\beta})^{-1}}}D^{-\frac{1}{2}}\\ &\leq G^{-1}3^{-\log r_0(\log\tilde{\beta})^{-1}}\max\left\{\tilde{C},\sqrt{\tilde{C}r_0^{-1}}\right\}r_k^{\frac{\log 3}{\log\tilde{\beta}}}D^{-\frac{1}{2}}\\ &\leq Hr^{\frac{\log 3}{\log\tilde{\beta}}}D^{-\frac{1}{2}}.\end{aligned}$$

This completes the proof. □

7 A maximum principle

In this section, we shall return to the region where r is sufficiently large, and with the help of equation (50), prove an estimate for θ, ϕ, and $\phi\lambda+\theta$ in the interior of a characteristic rectangle $\mathcal{X}$, in terms of certain bounds on $\partial\mathcal{X}$. This estimate, as we shall see, depends on what is essentially a *maximum principle*.

As with the previous section, the relevance of the assumptions of the proposition below will become clear in the context of our induction argument of Section 9.

Proposition 7.1. *Let $R>r_+$ be such that*

$$G_R^{-1}<2, \tag{96}$$

where G_R is the constant of Proposition 5.5 applied to $r_0=R$, assume

$$R\geq 8(e^2r_1^{-1}+M), \tag{97}$$

and let A and B be two nonnegative constants. Consider a characteristic rectangle $\mathcal{X}=[u_1,u_2]\times[v_1,v_2]$, and assume

$$r(u_2,v_1)=R,$$

$$|\theta|(u,v_1)\leq A \tag{98}$$

for all $u\in[u_1,u_2]$,

$$|\phi\lambda+\theta|(u_1,v)\leq B, \tag{99}$$

$$|\phi|(u_1,v)\leq A \tag{100}$$

for $v_1\leq v\leq v_2$, and

$$|\phi|(u,v_2)\leq A, \tag{101}$$

for all $u_1 \leq u \leq u_2$*. It follows that there exist constants* $\sigma_1, \sigma_2 > 0$*, depending only on* R*, such that*

$$|\phi|(u,v) \leq \sigma_1 \max\{A,B\}, \tag{102}$$

$$|\phi\lambda + \theta|(u,v) \leq B + \sigma_1 r^{-1}(u,v) \max\{A,B\}, \tag{103}$$

$$|\theta|(u,v) \leq \sigma_1 \max\{A,B\} \tag{104}$$

for all points $(u,v) \in \mathcal{X}$*, and*

$$\varpi(u_2,v_2) - \varpi(u_2,v_1) \leq \sigma_2 \left(\max\{A,B\}^2 + B^2 r(u_2,v_2)\right). \tag{105}$$

Proof. We first show that the bound

$$|\phi| \leq \sigma' \max\{A,B\} \tag{106}$$

holds throughout $\mathcal{X}$, for some sufficiently large constant $\sigma' > 1$. For this, define the number $u_* \geq u_1$ given by

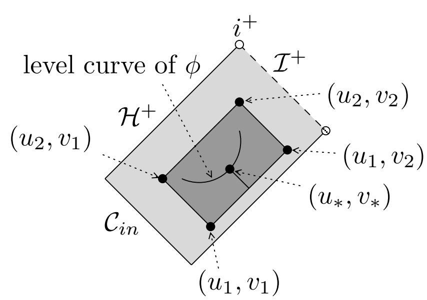
$$u_* = \sup_{u_1 \leq u \leq u_2} \{|\phi(u',v)| < \sigma' \max\{A,B\} \text{ for all } v \in [v_1,v_2], u' \in [u_1,u]\}.$$

(Note that if $\sigma' > 1$ then the set referred to on the right is non-empty, and u_* is in fact necessarily strictly greater that u_1 by continuity of ϕ.)

If $u_* = u_2$ then clearly (106) holds throughout $\mathcal{X}$, and we are done. Otherwise, $u_* < u_2$, and there exists $v_* \in [v_1, v_2]$ such that

$$|\phi(u_*,v_*)| = \sigma' \max\{A,B\}. \tag{107}$$

Consider first the possibility that $v_* \in (v_1,v_2)$:



Since $|\phi(u_*,v)| \leq \sigma' \max\{A,B\}$ for all $v \in [v_1,v_2]$, it follows that

$$\partial_v \phi(u_*,v_*) = 0.$$

Integrating now (50) on $[u_1,u_*] \times \{v_*\}$, we obtain that

$$\begin{aligned} |(\phi\lambda+\theta)(u_*,v_*)| &\leq B + \frac{2\sigma'(e^2 r_1^{-1} + M)}{r(u_*,v_*)} \max\{A,B\} \\ &\leq \frac{1}{2}\sigma' \max\{A,B\} \end{aligned} \tag{108}$$

where here we are using the fact that $r \geq R$, the condition (97), and, in addition, we are assuming that σ' has been chosen sufficiently large. Now we note that $\theta(u_*, v_*) = r^{-1}\partial_v \phi(u_*, v_*) = 0$, so

$$\begin{aligned} |\phi|(u_*, v_*) &= \lambda^{-1}|\phi\lambda + \theta| \\ &\leq \frac{1}{2}G^{-1}\sigma' \max\{A, B\} \\ &< \sigma' \max\{A, B\}, \end{aligned}$$

where, for the last inequality, we use that R satisfies (96). But this contradicts (107).

Since $v_* = v_2$ would contradict (101) for $\sigma' > 1$, we clearly must have $v^* = v_1$. Integrating (50) as above, we again obtain the bound (108) for $|\phi\lambda + \theta|(u_*, v_i)$. Thus, we have

$$\begin{aligned} |\phi(u_*, v_1)| &\leq \lambda^{-1}|\phi\lambda + \theta|(u_*, v_1) + \lambda^{-1}|\theta|(u_*, v_1) \\ &\leq G^{-1}\frac{1}{2}\sigma' \max\{A, B\} + G^{-1}A \\ &< \sigma' \max\{A, B\}, \end{aligned}$$

which again contradicts (107). (Note that to derive the second inequality above we use (98) and to derive the third we require σ' to be sufficiently large.) Thus we have established (106).

Integrating (50) from $u = u_1$, we obtain

$$|\phi\lambda + \theta|(u, v) \leq B + 2r^{-1}(e^2 r_1^{-1} + M)\sigma' \max\{A, B\}. \tag{109}$$

From (109), (103) follows for sufficiently large σ_1, as does (104), applying the triangle inequality, (106) and the bound $\lambda \leq G^{-1} \leq 2$.

Integrating the equation

$$\partial_v(r\phi) = \phi\lambda + \theta \tag{110}$$

along $\{u_2\} \times [v_1, v_2]$, we obtain

$$\begin{aligned} |r\phi(u_2, v)| &\leq R\sigma_1 \max\{A, B\} + BG_1^{-1}(r - R) \\ &\quad + 2M \log(r/R) G_1^{-1}\sigma' \max\{A, B\} && (111) \\ &\leq \sigma'' \max\{A, B\}(\log r) + Br && (112) \end{aligned}$$

and thus

$$|\phi(u_2, v)| \leq \sigma'' \left(\max\{A, B\} r^{-1} \log r + B\right). \tag{113}$$

In view now of (109) and (113) it follows by the triangle inequality and $\lambda \leq G^{-1}$ that

$$|\theta(u_2, v)| \leq \sigma''' \left(\max\{A, B\} r^{-1} \log r + B\right). \tag{114}$$

From (114), (27), and (56), we obtain finally the energy estimate

$$
\begin{aligned}
\varpi(u_2,v)-\varpi(u_2,v_1) &\leq \frac{1}{2}\hat{c}^{-1}\int_{v_i}^{v}\theta^2(u_2,v^*)dv^* \\
&\leq \frac{1}{2}\hat{c}^{-1}G_1^{-1}\sigma'''\Big(\max\{A,B\}^2 \\
&\qquad \cdot\int_{v_1}^{v} r^{-2}\log^2 r(u_1,v^*)\lambda dv^* + B^2 r(u_2,v)\Big) \\
&\leq \sigma_2\left(\max\{A,B\}^2+B^2 r(u_2,v)\right),
\end{aligned}
$$

for σ_2 suitably large. This yields (105). $\square$

8 Two more estimates

Proposition 8.1. *Let $r_* > r_0$ be given and consider a characteristic rectangle $\mathcal{Y}=[u_1,u_2]\times[v_1,v_2]\subset\mathcal{D}'$. Assume that for all $u\in[u_1,u_2]\cap\{r(u,v_2)\geq r_*\}$,*

$$|\phi|(u,v_2)\leq A, \tag{115}$$

and

$$\varpi(u_1,v_2)-\varpi(u_2,v_1)\leq A^2. \tag{116}$$

It follows that there exists a constant σ depending only on r_*, such that for $(u,v)\in\mathcal{Y}\cap\{r\geq r_*\}$

$$|\phi|(u,v)\leq \sigma A. \tag{117}$$

Assuming in addition that

$$\left|\frac{\zeta}{\nu}\right|(u,v_1)\leq A, \tag{118}$$

for all $u\in[u_1,u_2]\cap\{r(u,v_1)\leq r_*\}$, it follows that

$$\left|\frac{\zeta}{\nu}\right|(u,v)\leq \sigma A, \tag{119}$$

$$|\phi|(u,v)\leq \sigma A, \tag{120}$$

in $\mathcal{Y}\cap\{r\leq r_*\}$.

Proof. It follows from (40), (41), and (115) that the energy estimates

$$\frac{1}{2}\int_{v_1}^{v_2}\theta^2(u,v)\kappa^{-1}dv\leq A^2, \tag{121}$$

for $u_1\leq u\leq u_2$, and

$$\frac{1}{2}\int_{u_1}^{u_2}\left(\frac{\zeta}{\nu}\right)^2(1-\mu)(-\nu)(u,v)du\leq A^2,$$

for $v_1 \leq v \leq v_2$, hold.

Integrating the equation

$$\partial_v \phi = \frac{\theta}{r}$$

from $v = v_2$, i.e. from right to left, using the Schwarz inequality and the bound (115), we obtain that for $(u,v) \in \{r \geq r_*\}$

$$\begin{aligned} |\phi(u,v)| &\leq |\phi(u,v_2)| + \sqrt{\int_v^{v_2} \theta^2 \frac{1-\mu}{\lambda}(u,\tilde{v}) d\tilde{v}} \sqrt{\int_v^{v_2} \frac{\lambda}{1-\mu} \frac{1}{r^2}(u,\tilde{v}) d\tilde{v}} \\ &\leq A + \sqrt{2} A c^{-1} \sqrt{\int_v^{v_2} \frac{\lambda}{r^2}(u,\tilde{v}) d\tilde{v}} \\ &\leq \sigma A, \end{aligned}$$

and this yields (117).

Now consider the additional assumption (118). For $r \leq r_*$, we will use the "red-shift" technique. Integrating (51), we obtain

$$\begin{aligned} \left|\frac{\zeta}{\nu}\right|(u,v) &\leq \sqrt{\int_{v_1}^{v} \theta^2 \kappa^{-1}(u,\tilde{v}) d\tilde{v}} \sqrt{\int_{v_1}^{v} \frac{\kappa}{r^2} e^{-2\hat{c}\tilde{c}' r_*^{-2}(v-\tilde{v})} d\tilde{v}} \\ &\quad + \left|\frac{\zeta}{\nu}\right|(u,v_1) \\ &\leq \sigma' \sqrt{A^2} + A \\ &\leq \sigma A \end{aligned}$$

for sufficiently large $\sigma > 1$. Thus we have (119). Integrating the equation $\partial_u \phi = \frac{\zeta}{\nu}\frac{\nu}{r}$ from $r = r_*$, we immediately obtain (120). □

Proposition 8.2. *Suppose R is as in Proposition 7.1, and let $r(u_1, v_1) = R$. Consider a characteristic rectangle $\mathcal{Z} = [u_1, u_2] \times [v_1, v_2] \subset \mathcal{D}$. Suppose that*

$$|\phi| \leq A$$

on $\mathcal{Z} \cap \{r = R\}$, and

$$|\phi|(u, v_2) \leq A,$$

$$|\phi\lambda + \theta|(u_1, v) \leq B,$$

for $u_1 \leq u \leq u_2$ and $v_1 \leq v \leq v_2$, respectively. Then there exists a constant $\sigma > 1$ depending only on R such that

$$|\phi|(u,v) \leq \sigma \left(\max\{A,B\} r^{-1} R + B\right), \tag{122}$$

$$|\phi\lambda + \theta|(u,v) \leq \sigma \left(B + r^{-2} \log r \max\{A,B\}\right), \tag{123}$$

for all $(u,v) \in \mathcal{Z} \cap \{r \geq R\}$.

Proof. We first note that by integrating the equation $\partial_v(r\phi) = \phi\lambda + \theta$ along $\{u_1\} \times [v_1, v]$, we obtain

$$|r\phi(u_1, v)| \leq RA + G^{-1}(r(u_1, v) - R)B,$$

and thus

$$|\phi(u_1, v)| \leq \sigma' B + A$$

for σ' sufficiently large. Applying the argument of Proposition 7.1 we immediately obtain

$$|\phi(u, v)| \leq \sigma'' \max\{A, B\} \tag{124}$$

for all $(u, v) \in \mathcal{Z} \cap \{r \geq R\}$.

Integrating now (50) with the bound (124), we obtain

$$|\phi\lambda + \theta| \leq B + 2\sigma'' M \max\{A, B\} r^{-1}.$$

Integrating (110) with the above bound, from $r = R$, we obtain

$$|r\phi| \leq RA + B(r - R) + 2MG^{-1}\sigma'' \max\{A, B\} \log(r/R),$$

so

$$|\phi| \leq B + \sigma''' \max\{A, B\} r^{-1} \log r.$$

Once again integrating (50), we obtain

$$|\phi\lambda + \theta| \leq B + Br^{-1} + 2MG^{-1}\sigma''' r^{-2} \log(r/R) \max\{A, B\}.$$

This yields (123) for sufficiently large σ. Integrating yet again (110) yields

$$|r\phi| \leq AR + G^{-1}B(r - R) + G^{-1}B \log r + \sigma'''' \max\{A, B\}.$$

Dividing by r, we finally obtain

$$|\phi| \leq \sigma(B + r^{-1}R \max\{A, B\}),$$

for large enough σ. This completes the proof. $\Box$

9 Decay

In view of the estimates of the previous 3 sections, all the tools necessary to extract decay for ϕ and θ are in place. The glue binding these estimates together is the global geometry of $\mathcal{D}$: the relation between the v-length of suitably positioned characteristic rectangles and the r-difference between their vertices.

Before continuing, a brief word about our argument is in order. We will obtain decay by induction, each step improving the rate achieved in the previous. The 0'th step corresponds to the uniform boundedness that we have already proven. Note that at this 0'th step, we also have a bound for the total energy flux on $\mathcal{H}^+$, namely $\varpi_+ - \varpi_1$, and this is a (much) better bound than the bound

obtained for the energy flux by plugging in our pointwise bound for θ, namely $\int_1^\infty \hat{c}^{-1}\hat{r_0}^{-2}\hat{C}^2 = \infty$! The catalyst that allows the improvement at each step is the pigeon-hole principle applied to the energy flux on $\mathcal{H}^+$. The principle is quite simple: If the flux is small on a long interval, then every so often it is even smaller. We shall quantify this in the course of the proof by considering the event horizon decomposed essentially dyadically[32] into segments, and applying the pigeon hole principle in *each* dyadic interval to find a subinterval, of length $\sim v^p$ where $0 < p < 1$ and v is the v-coordinate of the endpoint, such that the flux in this subinterval is smaller than the flux in the original by a factor of v^{-1+p}. Starting from these "good" subintervals, we will construct rectangles $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, and, using the estimates proven previously, we will "spread" the better factor to all of $\mathcal{D}$, leaving us in particular with a pointwise bound for θ on $\mathcal{H}^+$, and a *better* bound on the flux than that which arises from integrating this pointwise bound on $\mathcal{H}^+$. We apply the pigeon-hole principle again and continue the induction.

We proceed now with the argument. Because it is complicated to keep track of different regions at the same time, we have split the argument into two main parts.

9.1 The induction: Part I

Define

$$\omega_* = \min\{\omega, 2\}$$

The result of this section is the following

Theorem 9.1. *There exist constants $C_{\omega_*-\frac{1}{2}} > 0$, $\delta_{\omega_*-\frac{1}{2}} > 0$, such that in the region $r \leq \delta_{\omega_*-\frac{1}{2}} v$, we have*

$$|\phi| \leq C_{\omega_*-\frac{1}{2}} \left(v^{-\left(\omega_*-\frac{1}{2}\right)} r^{-1} + v^{-\omega_*} \right), \tag{125}$$

$$|\theta| \leq C_{\omega_*-\frac{1}{2}} \left(v^{-\left(\omega_*-\frac{1}{2}\right)} r^{-1} + v^{-\omega_*} \right), \tag{126}$$

and in addition:

$$\varpi_+ - \varpi(\infty, v) \leq C_{\omega_*-\frac{1}{2}} v^{-2\left(\omega_*-\frac{1}{2}\right)}.$$

Proof. A series of real numbers w_k will be defined inductively. Given w_k, define the statement S_k as follows:

(S_k) There exist constants $\bar{C}_k, \delta_k > 0$, such that for $r \leq \delta_k v$,

$$|\phi| \leq \bar{C}_k \left(v^{-w_k} r^{-1} + v^{-\omega_*} \right), \tag{127}$$

$$|\theta| \leq \bar{C}_k \left(v^{-w_k} r^{-1} + v^{-\omega_*} \right), \tag{128}$$

$$\varpi_+ - \varpi(\infty, v) \leq \bar{C}_k v^{-2w_k}. \tag{129}$$

[32] i.e. such that the length of the segment is on the order of the v-coordinate of the endpoint

Choose a real number p satisfying

$$0 < p < \frac{1}{2(\alpha+1)+1} \tag{130}$$

where α is the constant of Proposition 6.3. The inductive step we shall prove is

Proposition 9.1. *S_k implies S_{k+1}, where w_{k+1} is defined by*

$$w_{k+1} = \min\left\{w_k + p, \omega_* - \frac{1}{2}\right\}. \tag{131}$$

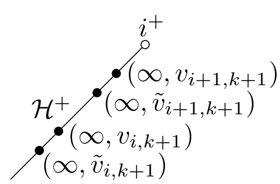
Assume for now Proposition 9.1 is true. Setting $w_0 = 0$, Proposition 5.3 implies that S_0 is true. It then follows by induction that S_k is true for all $k \geq 0$ where w_k is defined inductively by (131). Since $w_k = \omega_*$ for $k = [\frac{3}{2p}] + 1$, the theorem follows. $\Box$

Proof of Proposition 9.1. Fix a sufficiently large R satisfying Proposition 5.5 with constant G_R. Set $h_{k+1} > 1$ such that

$$4(h_{k+1}^2 - 1)(1 + G_R^{-1})h_{k+1} \leq \delta_k. \tag{132}$$

We have

Lemma 9.1.1. *Assume S_k holds for some $w_k > 0$. There exist a sequence of points $(\infty, v_{i,k+1}) \in \mathcal{H}^+$ and $(\infty, \tilde{v}_{i,k+1}) \in \mathcal{H}^+$,*



such that

$$\frac{1}{4h_{k+1}^p}(h_{k+1} - 1)v_{i,k+1}^p \leq v_{i,k+1} - \tilde{v}_{i,k+1} \leq (h_{k+1} - 1)v_{i,k+1}^p, \tag{133}$$

$$\frac{1 + h_{k+1}}{2} v_{i,k+1} \leq v_{i+1,k+1} \leq h_{k+1}^2 v_{i,k+1}, \tag{134}$$

$$\varpi(\infty, v_{i,k+1}) - \varpi(\infty, \tilde{v}_{i,k+1}) \leq 2h_{k+1}^{2w_k+1-p} \bar{C}_k v_{i,k+1}^{-2w_k-1+p}, \tag{135}$$

and

$$|\theta(\infty, v_{i,k+1})| \leq 8(h_{k+1} - 1)^{-1} r_+ (r_1^{-1} + 1) h_{k+1}^{p+w_k} \bar{C}_k v_{i,k+1}^{-w_k-p}. \tag{136}$$

Proof. Set $\hat{v}_{i,k+1} = h^i_{k+1}$. Consider the points $(\infty, \hat{v}_{i,k+1})$. Bisect the segment $\{\infty\} \times [\hat{v}_{i,k+1}, \hat{v}_{i+1,k+1}]$, and partition the "second" half $\{\infty\} \times [\frac{1}{2}(\hat{v}_{i,k+1} + \hat{v}_{i+1,k+1}), \hat{v}_{i+1,k+1}]$ into $[\hat{v}^{1-p}_{i,k+1}]$ subsegments of equal length $L_{i,k+1}$. We have[33]

$$[\hat{v}^{1-p}_{i,k+1}] \geq \frac{1}{2}\hat{v}^{1-p}_{i,k+1}, \tag{137}$$

$$(h_{k+1} - 1)\hat{v}^p_{i,k+1} \geq L_{i,k+1} \geq \frac{1}{2}(h_{k+1} - 1)\hat{v}^p_{i,k+1}. \tag{138}$$

Since–on account of (129)–we have

$$\varpi(\infty, \hat{v}_{i+1,k+1}) - \varpi(\infty, \hat{v}_{i,k+1}) \leq \bar{C}_k \hat{v}^{-2w_k}_{i,k+1},$$

it follows by (137) and the pigeon-hole principle that for each i, on one of the aforementioned subsegments of length $L_{i,k+1}$, call it $\{\infty\} \times [\tilde{v}_{i,k+1}, v''_{i,k+1}]$, we have

$$\varpi(\infty, v''_{i,k+1}) - \varpi(\infty, \tilde{v}_{i,k+1}) \leq 2\bar{C}_k \hat{v}^{-2w_k-1+p}_{i,k+1}. \tag{139}$$

Now bisect $\{\infty\} \times [\tilde{v}_{i,k+1}, v''_{i,k+1}]$, and consider again the "second half", i.e., the segment

$$\{\infty\} \times [v'_{i,k+1}, v''_{i,k+1}],$$

where

$$v'_{i,k+1} = \tilde{v}_{i,k+1} + \frac{1}{2}(v''_{i,k+1} - \tilde{v}_{i,k+1}).$$

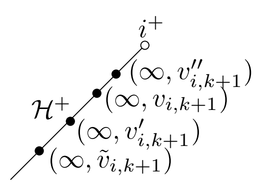
From (127) and the identity $\partial_v \phi = r^{-1}\theta$, we obtain

$$2\bar{C}_k(r_1^{-1} + 1)\hat{v}^{-w_k}_{i,k+1} \geq \left| \int_{v'_{i,k+1}}^{v''_{i,k+1}} \frac{\theta}{r}(\infty, v)dv \right|.$$

On the other hand, by (138) and the definition of $v'_{i,k+1}$, we have

$$v''_{i,k+1} - v'_{i,k+1} \geq \frac{1}{4}(h_{k+1} - 1)\hat{v}^p_{i,k+1},$$

so there must be a point in $\{\infty\} \times [v'_{i,k+1}, v''_{i,k+1}]$, call it $(\infty, v_{i,k+1})$:



where

$$|\theta|(\infty, v_{i,k+1}) \leq 8(h_{k+1} - 1)^{-1} r_+ (r_1^{-1} + 1)\bar{C}_k \hat{v}^{-w_k}_{i,k+1} \hat{v}^{-p}_{i,k+1}. \tag{140}$$

[33] Note that $v^{1-p}_{i,k+1}$ is not an integer.

By their construction, the points $v_{i,k+1}$ and $\tilde{v}_{i,k+1}$ satisfy (133) and (134). Inequality (140) together with the inequlity

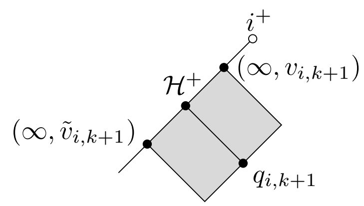
$$v_{i,k+1} \leq h_{k+1}^2 \hat{v}_{k+1} \tag{141}$$

yields (136). Similarly, we have by (139)

$$\begin{aligned} \varpi(\infty, v_{i,k+1}) - \varpi(\infty, \tilde{v}_{i,k+1}) &\leq \varpi(\infty, v''_{i,k+1}) - \varpi(\infty, \tilde{v}_{i,k+1}) \\ &\leq 2\bar{C}_k \hat{v}_{i,k+1}^{-2w_k-1+p}, \end{aligned}$$

which immediately yields (135), in view of (141). □

Fix now $r_0 > r_+$ and ϵ so that (88) of Proposition 6.1 holds, with $r_0 - r_+ < \frac{\epsilon}{2}$. For each i, bisect the segment $\{\infty\} \times [\tilde{v}_{i,k+1}, v_{i,k+1}]$, and consider the constant-v null segment passing through the midpoint. Denoting by $q_{i,k+1}$ the point on this ingoing segment where $r = r_0$, form the characteristic rectangle $\mathcal{W}_i = [u(q_{i,k+1}), \infty] \times [\tilde{v}_{i,k+1}, v_{i,k+1}]$:



Now let I_{k+1} be a positive integer so that

$$\tilde{v}_{I_{k+1},k+1} \geq v', \tag{142}$$

$$r_0 - r(\infty, \tilde{v}_{I_{k+1},k+1}) \leq \epsilon, \tag{143}$$

$$D \equiv (2\bar{C}_k)^{-1} h_{k+1}^{-2w_k-1+p} v_{I_{k+1},k+1}^{2w_k+1-p} > 1, \tag{144}$$

$$\begin{aligned} \log v_{I_{k+1},k+1} \quad \leq \quad & \frac{1}{(2w_k+1-p)} \left(\frac{(h_{k+1}-1) v_{I_{k+1},k+1}^p}{8 h_{k+1}^p} \right. \\ & \left. - \frac{r_0^2}{4\tilde{c}'\hat{c}} \log(\bar{C}_k 2 h_{k+1}(2w_k+1-p)) \right). \end{aligned} \tag{145}$$

Assume always in what follows that $i \geq I_{k+1}$. By (142), it follows that $\mathcal{W}_i \subset \mathcal{D}'$, by (143), it follows that (87) holds, by (145) and the inequality

$$v(q_i) - \tilde{v}_{i,k+1} \geq \frac{1}{4(h_{k+1}-1)} \hat{v}_{i,k+1}^p \geq \frac{1}{4(h_{k+1}-1) h_{k+1}^{2+p}} v_{i,k+1}^p, \tag{146}$$

it follows that (85) holds with D defined by (144), and finally, by (135), it follows that (86) holds. Thus, $\mathcal{W}_i$ satisfies the assumptions of Proposition 6.3. For $u \geq u(q_i)$, this gives the bound

$$\left|\frac{\zeta}{\nu}\right|(u, v_{i,k+1}) \leq \sqrt{2h_{k+1}^{2w_k+1-p}\bar{C}_k} H r^\alpha v_{i,k+1}^{-w_k-\frac{1}{2}+\frac{p}{2}}, \tag{147}$$

where H is the constant of (95). Integrating the equation

$$\partial_u \theta = -\frac{\zeta}{\nu}\frac{\nu}{r}\lambda$$

along $[u, \infty] \times \{v_{i,k+1}\}$, we obtain from (136) and (147) that

$$\begin{aligned} |\theta|(u, v_{i,k+1}) \quad &\leq \quad \alpha^{-1}\sqrt{2h_{k+1}^{2w_k+1-p}\bar{C}_k} H r^\alpha v_{i,k+1}^{-w_k-\frac{1}{2}+\frac{p}{2}} \\ &\quad + 8(h_{k+1}-1)^{-1} r_+ (r_1^{-1}+1) h_{k+1}^{p+w_k} \bar{C}_k v_{i,k+1}^{-w_k-p}, \end{aligned} \tag{148}$$

for all $u \geq u(q_i)$.

Now we note a fundamental fact: By our lower bound $|\lambda| \geq G_{r_0}$ of (77) in $r \geq r_0$, the inequality (146) and the equation

$$v_{i,k+1} - v(q_i) = v(q_i) - \tilde{v}_{i,k+1},$$

it follows that for

$$\tilde{\delta}_{k+1} = \frac{1}{4}(h_{k+1}-1) h_{k+1}^{2+p} G_{r_0}$$

we have

$$\{u : r(u, v_{i,k+1}) \leq \tilde{\delta}_{k+1} v_{i,k+1}^p\} \subset \{u \geq u(q_i)\}.$$

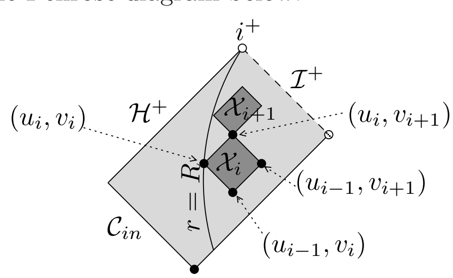
In other words, our rectangles $\mathcal{W}_i$ "penetrate" into the region where r is on the order of v to the positive power p. In particular, we have now that our bound (148) holds for all $(u, v_{i,k+1})$ with $r(u, v_{i,k+1}) \leq \tilde{\delta}_{k+1} v_{i,k+1}^p$.

To lighten the notation, let us denote in what follows $v_{i,k+1}$ by v_i. (We shall retain the k subscripts on all other constants, however.) Recall the constant R we have fixed at the beginning of the proof of this proposition. Let $\bar{I}_{k+1} \geq I_{k+1}$ be such that there exists a $u_{\bar{I}_{k+1}} \geq 1$ with $r(u_{\bar{I}_{k+1}}, v_{\bar{I}_{k+1}}) = R$. For $i \geq \bar{I}_{k+1}$, again there exists a unique u_i such that $r(u_i, v_i) = R$. In what follows, let us restrict to $i \geq \bar{I}_{k+1} + 1$.

Form the sequence of characteristic rectangles

$$\mathcal{X}_i = [u_{i-1}, u_i] \times [v_i, v_{i+1}]$$

as depicted in the Penrose diagram below:



It is clear that $r \geq R$ on $\bigcup_i \mathcal{X}_i$. Since

$$r(u_{i-1}, v_i) - r(u_i, v_i) = r(u_{i-1}, v_i) - r(u_{i-1}, v_{i-1}),$$

it follows from Proposition 5.5 that

$$u_i - u_{i-1} \leq G_R^{-1}(v_i - v_{i-1}) \leq G_R^{-2}(u_i - u_{i-1}).$$

Appealing to (134), we obtain that

$$\frac{G_R^2(h_{k+1}-1)}{2} v_i \leq r \leq (h_{k+1}^2 - 1)(1 + G_R^{-1})v_i + R \leq 2(h_{k+1}^2 - 1)(1 + G_R^{-1})v_i \tag{149}$$

along $\{u_{i-1}\} \times [v_i, v_{i+1}]$ and $[u_{i-1}, u_i] \times \{v_{i+1}\}$, where to show the final inequality we require that $i \geq \hat{I}_{k+1} \geq \bar{I}_{k+1} + 1$, for some constant $\hat{I}_{k+1}$. In what follows we shall assume now $i \geq \hat{I}_{k+1}$.

By (149) and (132), we have

$$\mathcal{X}_i \subset \{r \leq \delta_k v\}. \tag{150}$$

Thus, (127) and (128) apply in $\mathcal{X}_i$. Our first task is to obtain a bound for θ on $[u_{i-1}, u_i] \times \{v_i\}$ by interpolating (128) with (148).[34] For this we will apply the following

Lemma 9.1.2. *Let $\tilde{C}, w, x, y, z, \tilde{\delta}, p$ be positive real numbers, and let $v_0 \geq 1$. Suppose for all u such that $r(u, v_0) \leq \tilde{\delta} v^p$, we have*

$$|\theta|(u, v_0) \leq \tilde{C}(r^\alpha v_0^{-y} + v_0^{-z}), \tag{151}$$

and for some (u_0, v_0) we have

$$|\theta|(u_0, v_0) \leq \tilde{C}(r^{-1} v_0^{-w} + v_0^{-x}). \tag{152}$$

Then it follows that

$$|\theta|(u_0, v_0) \leq \tilde{C}(3 + \tilde{\delta}^{-p}) v_0^{-\tilde{w}}, \tag{153}$$

where

$$\tilde{w} = \min\left\{ \frac{y}{\alpha+1} + w\left(1 - \frac{1}{\alpha+1}\right), w + p, z, x \right\}.$$

Proof. If

$$r^\alpha v_0^{-y} \geq v_0^{-w} r^{-1}$$

then

$$r \geq v_0^{\frac{1}{1+\alpha}(y-w)},$$

[34] Remember that (148) does *not* apply on the whole segment.

and thus, by (152), we have

$$
\begin{aligned}
|\theta|(u_0,v_0) &\le \tilde{C}v_0^{-w}r^{-1}+\tilde{C}v_0^{-x} \\
&\le \tilde{C}v_0^{-w-\frac{1}{1+\alpha}(y-w)}+\tilde{C}v_0^{-x} \\
&= 2\tilde{C}v_0^{-\min\left\{\frac{y}{(\alpha+1)}+w\big)1-\frac{1}{\alpha+1}\big),x\right\}}, \qquad (154)
\end{aligned}
$$

while if

$$r^{\alpha}v_0^{-y}\le v_0^{-w}r^{-1},$$

then

$$r\le v_0^{\frac{1}{1+\alpha}(y-w)},$$

and thus by (151) and (152), we have

$$
\begin{aligned}
|\theta|(u_0,v_0) &\le \tilde{C}r^{\alpha}v_0^{-y}+\tilde{C}\tilde{\delta}^{-p}v_0^{-w-p}+\tilde{C}v_0^{-z}+\tilde{C}v_0^{-x} \\
&\le \tilde{C}v_0^{-y+\frac{\alpha}{\alpha+1}(y-w)}+\tilde{C}\tilde{\delta}^{-p}v_0^{-w-p}+\tilde{C}v_0^{-z}+\tilde{C}v_0^{-x} \\
&= \tilde{C}(3+\tilde{\delta}^{-p})v_0^{-\min\left\{\left(\frac{y}{(1+\alpha)}+w\left(1-\frac{1}{1+\alpha}\right)\right),w+p,x,z\right\}}. \qquad (155)
\end{aligned}
$$

Inequalities (154) and (155) together yield the result of the lemma. □

In view of (130), applying Lemma 9.1.2, with $w=w_k$, $z=w_k+p$, $y=w_k+\frac{1}{2}-\frac{p}{2}$, $x=\omega_*$, and $\tilde{\delta}=\tilde{\delta}_{k+1}$, it follows that on $[u_{i-1},u_i]\times\{v_i\}$, we have

$$|\theta|\le E_{k+1}v_i^{-w_{k+1}},$$

where

$$E_{k+1}=(3+\tilde{\delta}_{k+1}^{-p})\max\left\{\alpha^{-1}\sqrt{2h_{k+1}^{2w_k+1-p}\bar{C}_k}H,8(h_{k+1}-1)^{-1}r_+(r_1^{-1}+1)h_{k+1}^{p+w_k}\bar{C}_k\right\}.$$

We can also bound $|\phi|$ and $|\phi\lambda+\theta|$ on part of $\partial\mathcal{X}$. In view of (150), it follows from S_k and (149) that on $[u_{i-1},u_i]\times\{v_{i+1}\}$,

$$|\phi|\le 2G_R^{-2}(h_{k+1}-1)^{-1}\bar{C}_k(v_i^{-w_k-1}+v_i^{-\omega_*}), \qquad (156)$$

while on $\{u_{i-1}\}\times[v_i,v_{i+1}]$,

$$|\phi|\le 2G_R^{-2}(h_{k+1}-1)^{-1}\bar{C}_k(v_i^{-w_k-1}+v_i^{-\omega_*}), \qquad (157)$$

$$|\phi\lambda+\theta|\le 2^{\omega_*}G^{-2\omega_*}(h_{k+1}^2-1)^{-\omega_*}\hat{C}v_i^{-\omega_*}, \qquad (158)$$

where to obtain the latter inequality we use inequality (73). Since $w_{k+1}<\omega_*$ and $w_k+1>w_{k+1}$, it follows that we can apply Proposition 7.1 on $\mathcal{X}_i$ with

$$A=\max\{E_{k+1},2G^{-2}(h_{k+1}-1)^{-1}\bar{C}_k\}v_i^{-w_{k+1}},$$

$$B=2^{\omega_*}G^{-2\omega_*}(h_{k+1}^2-1)^{-\omega_*}\hat{C}v_i^{-w_{k+1}},$$

to obtain

$$|\phi| \leq \tilde{E}_{k+1} v_i^{-w_{k+1}}, \tag{159}$$
$$|\theta| \leq \tilde{E}_{k+1} v_i^{-w_{k+1}},$$

$$|\lambda\phi + \theta| \leq \tilde{E}_{k+1} \left(v_i^{-\omega_*} + r^{-1} v_i^{-w_{k+1}} \right) \tag{160}$$

in $\mathcal{X}_i$, and

$$\varpi(u_i, v_{i+1}) - \varpi(u_i, v_i) \leq \tilde{E}_{k+1}^2 \left(v_i^{-2w_{k+1}} + r(u_i, v_{i+1}) v_i^{-2\omega_*} \right), \tag{161}$$

where

$$\begin{aligned} \tilde{E}_{k+1} \quad = \quad & \max\{\sigma_1, \sqrt{\sigma_2}\} \cdot \max\left\{ E_{k+1}, 2G^{-2}(h_{k+1} - 1)^{-1} \bar{C}_k, \right. \\ & \left. 2^{\omega_*} G^{-2\omega_*} (h_{k+1}^2 - 1)^{-\omega_*} \hat{C} \right\}. \end{aligned}$$

Since $w_{k+1} \leq \omega_* - \frac{1}{2}$, (161) and (149) give the bound

$$\varpi(u_i, v_{i+1}) - \varpi(u_i, v_i) \leq (1 + 2(h_{k+1}^2 - 1)(1 + G_R^{-1})) \tilde{E}_{k+1}^2 v_i^{-2w_{k+1}}. \tag{162}$$

We can now easily obtain $(129)_{k+1}$: By (147), it follows that we have a bound

$$\varpi(u_i, v_i) - \varpi(\infty, v_i) \leq 2 h_{k+1}^{2w_k + 1 - p} \bar{C}_k H^2 (2\alpha + 1)^{-1} R^{2\alpha+1} v_{i,k+1}^{-2w_k - 1 + p},$$

and thus, by (162), a bound

$$\varpi(u_i, v_{i+1}) - \varpi(\infty, v_i) \leq \hat{E}_{k+1} v_i^{-2w_{k+1}}, \tag{163}$$

where

$$\hat{E}_{k+1} = 2 h_{k+1}^{2w_k + 1 - p} \bar{C}_k H^2 (2\alpha + 1)^{-1} R^{2\alpha+1} + (1 + 2(h_{k+1}^2 - 1)(1 + G_R^{-1})) \tilde{E}_{k+1}^2.$$

In particular,

$$\varpi(\infty, v_{i+1}) - \varpi(\infty, v_i) \leq \varpi(u_i, v_{i+1}) - \varpi(\infty, v_i) \leq \hat{E}_{k+1} v_i^{-2w_{k+1}},$$

so

$$\begin{aligned} \varpi_+ - \varpi(\infty, v_i) \quad & \leq \quad \sum_{j \geq i} (\varpi(\infty, v_{j+1}) - \varpi(\infty, v_j)) \\ & \leq \quad \hat{E}_{k+1} \sum_{j \geq i} v_j^{-2w_{k+1}} \\ & \leq \quad \hat{E}_{k+1} \sum_{j \geq i} \hat{v}_{j,k+1}^{-2w_{k+1}} \\ & \leq \quad \hat{E}_{k+1} (1 - h_{k+1}^{-i2w_{k+1}})^{-1} \hat{v}_{i,k+1}^{-2w_{k+1}}. \end{aligned}$$

For $v \geq v_{\hat{I}_{k+1}}$, let i be such that $v_i \leq v \leq v_{i+1}$. We have

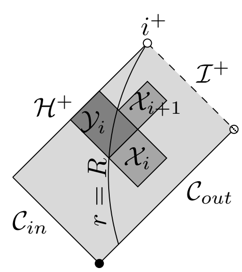
$$\begin{aligned}
\varpi_+ - \varpi(\infty, v) &\leq \varpi_+ - \varpi(\infty, v_i) \\
&\leq \hat{E}_{k+1} \left(1 - h_{k+1}^{-i2w_{k+1}}\right)^{-1} \hat{v}_{i,k+1}^{-2w_{k+1}} \\
&\leq \hat{E}_{k+1} h_{k+1}^{4w_{k+1}} \left(1 - h_{k+1}^{-\hat{I}_{k+1}2w_{k+1}}\right)^{-1} v^{-2w_{k+1}}.
\end{aligned}$$

Thus, for

$$\bar{C}_{k+1} \geq \max\left\{h_{k+1}^{4w_{k+1}} \left(1 - h_{k+1}^{-\hat{I}_{k+1}2w_{k+1}}\right)^{-1} \hat{E}_{k+1}, (\varpi_+ - \varpi_1) h_{k+1}^{\hat{I}_{k+1}+1}\right\}$$

we have $(129)_{k+1}$.

We still need to obtain our pointwise bounds $(127)_{k+1}$ and $(128)_{k+1}$. Form the rectangles $\mathcal{Y}_i = [u_i, \infty] \times [v_i, v_{i+1}]$:



From (163) and the pointwise bounds (147) and (159), it follows that the assumptions of Proposition 8.1 hold for $\mathcal{Y}_i$, with $r_* = R$ and

$$A = \sqrt{2h_{k+1}^{2w_k+1-p}\bar{C}_k H^2 R^{2\alpha+1} + (1 + 2(h_{k+1}^2 - 1)(1 + G_R^{-1}))\tilde{E}_{k+1}^2} v_{i,k+1}^{-w_{k+1}}.$$

Thus

$$|\phi| \leq \bar{E}_{k+1} v_i^{-w_{k+1}}, \tag{164}$$

$$\left|\frac{\zeta}{\nu}\right| \leq \bar{E}_{k+1} v_i^{-w_{k+1}}, \tag{165}$$

in $\mathcal{Y}_i$, where

$$\bar{E}_{k+1} = \sqrt{2h_{k+1}^{2w_k+1-p}\bar{C}_k H^2 R^{2\alpha+1} + (1 + 2(h_{k+1}^2 - 1)(1 + G_R^{-1}))\tilde{E}_{k+1}^2}.$$

Form now the sequence of rectangles $\mathcal{Z}_i = [u_i, u_{i+2}] \times [v_i, v_{i+1}]$. By (158) and (164), it follows that the $\mathcal{Z}_i$ satisfy the assumptions of Proposition 8.2, with $A = \bar{E}_{k+1} v_i^{-w_{k+1}}$, and $B = 2^{\omega_*} G^{-2\omega_*} (h_{k+1}^2 - 1)^{-\omega_*} \hat{C} v_i^{-\omega_*}$. Thus, we have

$$|\phi| \leq \tilde{C}_{k+1} \left(r^{-1} v_i^{-w_{k+1}} + v_i^{-\omega_*}\right), \tag{166}$$

$$|\lambda\phi+\theta|\leq\tilde{C}_{k+1}\left(r^{-2}(\log r)v_i^{-w_{k+1}}+v_i^{-\omega_*}\right) \tag{167}$$

in $\mathcal{Z}_i\cap\{r\geq R\}$, where

$$\tilde{C}_{k+1}=\sigma\max\left\{\bar{E}_{k+1},2^{\omega_*}G_R^{-2\omega_*}(h_{k+1}^2-1)^{-\omega_*}\hat{C}\right\}.$$

The inequalities (166) and (167), together with the triangle inequality and the bound $1\leq\lambda^{-1}\leq G$, immediately yield

$$|\theta|\leq 2G_R^{-1}\tilde{C}_{k+1}\left(r^{-1}v_i^{-w_{k+1}}+v_i^{-\omega_*}\right) \tag{168}$$

in $\mathcal{Z}_i\cap\{r\geq R\}$. Finally, integrating the equation

$$\partial_u\theta=-\frac{\zeta}{\nu}\frac{\nu}{r}\lambda$$

in $\mathcal{Y}_i$, from $r=R$, using the bound (165) and the initial bound provided by (168), yields the inequality

$$|\theta|\leq(2G_R^{-1}\tilde{C}_{k+1}(1+R^{-1})+\bar{E}_{k+1}\log(R/r_1))v_i^{-w_{k+1}} \tag{169}$$

in $\mathcal{Y}_i$.

Define the region

$$\mathcal{R}=\bigcup_{i\geq\hat{I}_k}(\mathcal{X}_i\cup\mathcal{Y}_i).$$

Given a $(u,v)\in\mathcal{R}$, setting i such that $v_i\leq v\leq v_{i+1}$, we have by (164) and (166)

$$\begin{aligned}|\phi(u,v)| &\leq \max\{R\bar{E}_{k+1},\tilde{C}_{k+1}\}\left(r^{-1}v_i^{-w_{k+1}}+v_i^{-\omega_*}\right)\\ &\leq \max\{R\bar{E}_{k+1},\tilde{C}_{k+1}\}h_{k+1}^{2\omega_*}\left(r^{-1}v^{-w_{k+1}}+v^{-\omega_*}\right).\end{aligned}$$

On the other hand, by (168) and (169) we have the bound

$$\begin{aligned}|\theta(u,v)| &\leq (2G_R^{-1}\tilde{C}_{k+1}(1+R^{-1})+\bar{E}_{k+1}R\log(R/r_1))\left(r^{-1}v_i^{-w_{k+1}}+v_i^{-\omega_*}\right)\\ &\leq (2G_R^{-1}\tilde{C}_{k+1}(1+R^{-1})+\bar{E}_{k+1}R\log(R/r_1))h_{k+1}^{2\omega_*}\left(r^{-1}v^{-w_{k+1}}+v^{-\omega_*}\right).\end{aligned}$$

For

$$\delta_{k+1}\leq 2(h_{k+1}^2-1)(1+G_R^{-1})h_{k+1}^{-2}$$

it follows by (149) that

$$\{r\leq\delta_{k+1}v\}\cap\{v\geq v_{\hat{I}_k}\}\subset\mathcal{R}.$$

The bound $(127)_{k+1}$ then follows for

$$\bar{C}_{k+1}\geq\max\left\{\left(\sup_{v\leq v_{\hat{I}_k}}|\phi|\right)\left(v_{\hat{I}_k}^{w_k}r(1,\hat{v}_{\tilde{I}_k})+v_{\tilde{I}_k}^{\omega_*}\right),h_{k+1}^{2\omega_*}R\bar{E}_{k+1},h_{k+1}^{2\omega_*}\tilde{C}_{k+1}\right\}$$

and $(128)_{k+1}$ follows for

$$\begin{aligned}\bar{C}_{k+1} \quad &\geq \quad \max\left\{\left(\sup_{v\leq v_{\hat{I}_k}}|\theta|\right)\left(v_{\hat{I}_k}^{w_k} r(1,\hat{v}_{\tilde{I}_k})+v_{\tilde{I}_k}^{\omega_*}\right),\right.\\ &\left.(2G_R^{-1}\tilde{C}_{k+1}(1+R^{-1})+\bar{E}_{k+1}R\log(R/r_1))h_{k+1}^{2\omega_*}\right\}.\end{aligned}$$

This completes the proof of the proposition. □

9.2 Decay on $\mathcal{I}^+$

So far we have only obtained decay in the region $r \leq \delta_{\omega_*-\frac{1}{2}}v$. In this section we will prove decay in $r \geq \delta_{\omega_*-\frac{1}{2}}v$, in particular, decay along $\mathcal{I}^+$. This decay will now be measured in the u direction.

Theorem 9.2. *There exists a constant $\tilde{C} > 0$ such that in $r \geq \delta_{\omega_*-\frac{1}{2}}v$, we have the bounds*

$$|\theta| \leq \tilde{C}v^{-1}u^{-(\omega_*-1)}, \tag{170}$$

$$|\zeta| \leq \tilde{C}u^{-\left(\omega_*-\frac{1}{2}\right)},$$

$$|r\phi| \leq \tilde{C}u^{-(\omega_*-1)}. \tag{171}$$

Proof. Fix $0 < \delta' < 1$ and consider the curve Γ defined by $u = \delta' v$. For δ' sufficiently close to 1, Γ is clearly contained in $r \leq \delta_{3/2}v$. This follows immediately by noting, on the one hand, that $r(1,v)v^{-1} \to 1$ as $v \to \infty$, while, on the other hand, if $r(u,v) \geq R$, then $r(1,v) - r(u,v) \geq G_R^{-1}u$, and $G_R^{-1} \to 1$ as $R \to \infty$. Choosing such a δ', we have that on Γ, $r \geq h(1-\delta')v$, for some $h > 0$. Thus, it follows from Theorem 9.1 that (171) holds on Γ where $\tilde{C}$ is replaced by a constant we shall call $\tilde{C}'$.[35]

If (u,v) is such that $u \leq \delta' v$, then we can integrate (110) along $\{u\}\times[\delta'^{-1}u, v]$ to obtain

$$\begin{aligned}|r\phi|(u,v) \quad &\leq \quad \tilde{C}'u^{-(\omega_*-1)} + \int_{\delta'^{-1}u}^{v}\hat{C}r^{-\omega_*}(u,\bar{v})d\bar{v}\\ &\leq \quad \tilde{C}'u^{-(\omega_*-1)} + G^{-1}\hat{C}r^{-(\omega_*-1)}(u,\delta'^{-1}u),\end{aligned} \tag{172}$$

which yields (171) for $\tilde{C}$ sufficiently large. We then obtain

$$\begin{aligned}|\theta|(u,v) \quad &\leq \quad G^{-1}|\phi\lambda+\theta|(u,v) + r^{-1}|r\phi|(u,v)\\ &\leq \quad \tilde{C}'\left(v^{-\omega_*} + v^{-1}u^{-(\omega_*-1)}\right),\end{aligned} \tag{173}$$

and this yields (170), again for sufficiently large $\tilde{C}$. Finally, we note that in the proof of Theorem 9.1, we have in fact shown that for $r(u,v_*) = r_0$,

$$|\zeta|(u,v_*) \leq \tilde{C}'v_*^{-\left(\omega_*-\frac{1}{2}\right)},$$

[35]In the remainder of this paper, we leave the explicit computation of constants to the reader.

for some $\tilde{C}'$. Integrating (30) using (126), we obtain the bound

$$\begin{aligned}
|\zeta|(u,\delta'^{-1}u) &\leq \tilde{C}' v_*^{-(\omega_*-\frac{1}{2})} + \int_{v_*}^{\delta'^{-1}u} \frac{|\theta\nu|}{r}(u,\bar{v})d\bar{v} \\
&\leq \tilde{C}' v_*^{-(\omega_*-\frac{1}{2})} \\
&\quad + C_{\omega_*-\frac{1}{2}} G^{-1}\left(r_0^{-1} v_*^{-(\omega_*-\frac{1}{2})} + \log(r(u,\delta'^{-1}u)/r_0) v_*^{-\omega_*}\right) \\
&\leq \tilde{C}'' v_*^{-(\omega_*-\frac{1}{2})}.
\end{aligned}$$

Now let $v \geq \delta'^{-1}u$. Integrating from Γ using our bound (173) for θ we obtain

$$\begin{aligned}
|\zeta|(u,v) &\leq \tilde{C}'' u^{-(\omega_*-\frac{1}{2})} + \int_{\delta'^{-1}u}^{v} \frac{|\theta\nu|}{r}(u,\bar{v})d\bar{v} \\
&\leq \tilde{C}'' u^{-(\omega_*-\frac{1}{2})} + \tilde{C}'''\left(r^{-\omega_*} + u^{-\omega_*-1} r^{-1}(u,\delta'^{-1}u)\right) \\
&\leq \tilde{C} u^{-(\omega_*-\frac{1}{2})}
\end{aligned}$$

for $\tilde{C}$ sufficiently large. This completes the proof. □

We can use the above theorem to obtain a better decay estimate in r for the quantity $\phi\lambda + \theta$:

Proposition 9.2. *There exists a constant $\hat{C}'$ such that*

$$|\phi\lambda + \theta| \leq \hat{C}'(r^{-3}\log r + r^{-\omega}). \tag{174}$$

Proof. The inequality (174) holds on the initial outgoing null curve. Integrating (50) using (171), we obtain immediately (174) in the region $r \geq \delta_{\omega_*-\frac{1}{2}} v$. Using (125), the estimate extends to $r \leq \delta_{\omega_*-\frac{1}{2}} v$. □

We will remove the log term in (174) in the context of the proof of the following theorem. The r^{-3} term cannot be improved, however, because it arises independently of the rate of the u-decay of $|r\phi|$. This is ultimately the source of the exponent 3 in Price's law, and is the reason why we have restricted consideration to $\omega \leq 3$ in Assumption $\mathbf{\Delta}'$.

We noted immediately before the statement of Theorem 9.1 in Section 9 that the obstruction at ω_* arose from the bound (73) applied in (160). As this bound has now been improved, we can improve Theorem 9.1 with the following

Theorem 9.3. *There exist constants $C_{\omega-\frac{1}{2}} > 0$, $\delta_{\omega-\frac{1}{2}} > 0$ such that in the region $r \leq \delta_{\omega-\frac{1}{2}} v$, we have*

$$|\phi| \leq C_{\omega-\frac{1}{2}}\left(v^{-(\omega-\frac{1}{2})} r^{-1} + v^{-\omega}\right),$$

$$|\theta| \leq C_{\omega-\frac{1}{2}}\left(v^{-(\omega-\frac{1}{2})} r^{-1} + v^{-\omega}\right),$$

and in addition

$$\varpi_+ - \varpi(\infty, v) \leq C_{\omega-\frac{1}{2}} v^{-(2\omega-1)}.$$

Proof. We first show the above where $\omega < 3$. For this, one proceeds exactly as in the proof of Theorem 9.1, except that we now define

$$w_{k+1} = \min\{w_k + p, s\},$$

for $p > 0$ defined as before. Inequality (174) implies $|\phi\lambda + \theta| \leq C(3-\omega)r^{-\omega}$, and thus the $v_i^{-\omega_*}$ in (156), (157), and (160) can be replaced by $v_i^{-s-\omega}$, and, in (161), the $v_i^{-2\omega_*}$ can be replaced by $v_i^{-2\omega}$. Finally, since $w_k \leq \frac{5}{2}$, (162) holds as well, and the rest follows as before.

If $\omega = 3$, then first carry through the argument of the above paragraph with ω replaced with some ω' satisfying $2 < \omega' < \omega$. One improves Proposition 9.2 to obtain in particular $|r\phi| \leq \tilde{C}\left(u^{-(\omega'-\frac{1}{2})} + u^{-2}\right)$ for some constant $\tilde{C}$. ¿From this one improves (174) to obtain the uniform decay in r

$$|\phi\lambda + \theta| \leq \hat{C}' r^{-3}. \tag{175}$$

for sufficiently large $\hat{C}'$. But now one can carry through the argument of the previous paragraph with ω. □

9.3 The induction: Part II

Theorem 9.3 removes the obstruction of (73) in (160). The obstruction at $v^{-\left(\omega-\frac{1}{2}\right)}$, on the other hand, arises the *second* term of (105) from Proposition 7.1, i.e. in the second term of (161). To circumvent this obstruction, we must apply the propositions of Section 7 and 8 to rectangles $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ whose v-dimensions are smaller, i.e. so that $r(u_i, v_{i+1}) \sim v_{i+1}^s$ with $0 < s < 1$. We proceed to do this in the present section.

Theorem 9.4. *Let $\epsilon > 0$. Then there exist constants $C_{\omega-\epsilon} > 0$, $\delta_{\omega-\epsilon} > 0$ such that for $r \leq \delta_{w-\epsilon} v$, we have*

$$|\phi| \leq C_{\omega-\epsilon}\left(v^{-(\omega-\epsilon)}r^{-1} + v^{-\omega}\right),$$

$$|\theta| \leq C_{\omega-\epsilon}\left(v^{-(\omega-\epsilon)}r^{-1} + v^{-\omega}\right),$$

$$|\phi\lambda + \theta| \leq C_{\omega-\epsilon}\left(v^{-(\omega-\epsilon)}r^{-1} + v^{-\omega}\right).$$

Proof. Consider the statement S'_τ

(S'_τ) There exist constants constants $\bar{C}_\tau, \delta_\tau > 0$, such that for $r \leq \delta_\tau v$,

$$|\phi| \leq \bar{C}_\tau\left(v^{-(\omega-\tau)}r^{-1} + v^{-\omega}\right), \tag{176}$$

$$|\theta| \leq \bar{C}_\tau\left(v^{-(\omega-\tau)}r^{-1} + v^{-\omega}\right), \tag{177}$$

$$|\lambda\phi + \theta| \leq \bar{C}_\tau\left(v^{-(\omega-\tau)}r^{-1} + v^{-\omega}\right),$$

$$\varpi(v + v^\tau) - \varpi(v) \leq \bar{C}_\tau v^{-2(\omega-\tau)}.$$

We will prove the following

Proposition 9.3. *S'_τ implies $S'_{\frac{\tau}{2}}$.*

Assuming for the moment the above Proposition, and given any $\epsilon > 0$, there exists an n such that $\omega - \frac{1}{2^n} > \omega - \epsilon$. Since $S'_{\frac{1}{2}}$ is true by Theorems 9.1 and 9.3, then by induction, $S'_{\frac{1}{2^n}}$ is true. But $S'_{\frac{1}{2^n}}$ immediately implies the theorem. $\Box$.

Proof of Proposition 9.3. This we shall also prove by induction! Consider the statement S''_k defined by

(S''_k) There exist constants $\bar{C}_k, \delta_k > 0$, such that for $r \leq \delta_k v$,

$$|\phi| \leq \bar{C}_k \left(v^{-w_k} r^{-1} + v^{-\omega}\right), \tag{178}$$

$$|\theta| \leq \bar{C}_k \left(v^{-w_k} r^{-1} + v^{-\omega}\right), \tag{179}$$

$$|\phi\lambda + \theta| \leq \bar{C}_k \left(v^{-w_k} r^{-1} + v^{-\omega}\right), \tag{180}$$

$$\varpi(v + v^\tau) - \varpi(v) \leq \bar{C}_k v^{-2w_k}. \tag{181}$$

Choose a p such that

$$0 < p < \frac{\tau}{2\alpha + 3}. \tag{182}$$

We will prove

Proposition 9.4. *S''_k implies S''_{k+1} where*

$$w_{k+1} = \min\left\{w_k + p, \omega - \frac{\tau}{2}\right\}.$$

Assuming the result of Proposition 9.4, Proposition 9.3 follows immediately by induction. For setting $w_0 = \omega - \tau$, S''_0 is true, as it is implied by S'_τ, and thus by induction S''_k is true for all $k \geq 0$. There clearly exists a k such that $w_k = \omega - \frac{\tau}{2}$, and for such a k, S''_k immediately gives $S'_{\frac{\tau}{2}}$. $\Box$

Proof of Proposition 9.4. We fix $h_{k+1} > 0$ sufficiently small, set $\hat{v}_{0,k+1} = 1$, and define inductively

$$\hat{v}_{i+1,k+1} = \hat{v}_{i,k+1} + h_{k+1} \hat{v}^\tau_{i,k+1}.$$

In analogy with the proof of Theorem 9.1, we obtain a sequence of segments $\{\infty\} \times [\tilde{v}_{i,k+1}, v_{i,k+1}]$, such that $v_{i,k+1} \sim \hat{v}_{i,k+1}$ and

$$\varpi(v_{i,k+1}) - \varpi(\tilde{v}_{i,k+1}) \leq \tilde{C} v_{i,k+1}^{-2w_k - \tau + p},$$

$$|\theta|(\infty, v_{i,k+1}) \leq \tilde{C} v_{i,k+1}^{-w_k + \frac{p}{2} - \frac{\tau}{2}}.$$

Constructing as before characteristic rectangles $\mathcal{W}_i$, and applying Proposition 6.3, we obtain that there exist constants $\tilde{C}'$ and $\delta > 0$ such that, for $r \le \delta v_i^p$, we have

$$|\theta(u, v_i)| \le \tilde{C}' \left(v_i^{-w_k - p} + r^\alpha v_i^{-w_k + \frac{p}{2} - \frac{\tau}{2}} \right).$$

Again, we have here dropped the $k+1$ subscript from $v_{i,k+1}$. On the other hand, from our assumption S_k'', we have that for $r(u, v_i) \le \delta_k v$,

$$|\theta(u, v_i)| \le \bar{C}_k \left(v_i^{-w_k} r^{-1} + v_i^{-\omega} \right).$$

Applying Lemma 9.1.2 with $w = w_k$, $x = \omega$, $y = w_k + \frac{\tau}{2} - \frac{p}{2}$, $z = w_k + p$, in view of our choice (182), it follows that for $r \le \delta v_i$, if δ is chosen sufficiently small, we have

$$|\theta(u, v_i)| \le \tilde{C}' v_i^{-w_{k+1}},$$

for some $\tilde{C}'$. We then construct rectangles $\mathcal{X}_i$, $\mathcal{Y}_i$ from the sequence v_i as before, arranging by appropriate choice of h_{k+1} for $\bigcup_{i \ge I} \mathcal{X}_i$ to be contained in the region $r \le \delta_k v$ for sufficiently large I. On the other hand, $\bigcup_{i \ge I} (\mathcal{X}_i \cup \mathcal{Y}_i)$ covers the region $r \le \delta_{k+1} v^\tau$ for small enough $\delta_{k+1} < \delta_k$. Since $w_k \ge \omega - \tau$, in the region $\delta_{k+1} v^\tau \le r \le \delta_{k+1} v$, $(178)_k$, $(179)_k$, $(180)_k$ immediately yield $(178)_{k+1}$, $(179)_{k+1}$, and $(180)_{k+1}$, since in that region, the second term of these inequalities always dominates. Thus, to show the proposition, it suffices to restrict to $\bigcup_{i \ge I} (\mathcal{X}_i \cup \mathcal{Y}_i)$. The assumptions of Proposition 6.1 clearly apply to $\mathcal{X}_i$ with $A = \tilde{C}' v_i^{-w_{k+1}}$, $B = \tilde{C}' v_i^{-\omega}$, after redefinition of the constant $\tilde{C}'$. Since $2\omega - \tau \ge 2w_{k+1}$, it follows that the assumptions of Proposition 6.2 then apply to the rectangles $\mathcal{Y}_i$ with $A = \tilde{C}' v_i^{-w_{k+1}}$. In particular this yields $(181)_{k+1}$. Finally, the assumptions of Proposition 6.3 apply to the rectangles $\mathcal{Z}_i$ with $A = \tilde{C}' v_i^{-w_{k+1}}$ and $B = \tilde{C}' v_i^{-\omega}$, again after redefinition of $\tilde{C}'$. All together, the bounds obtained yield $(178)_{k+1}$, $(179)_{k+1}$, and $(180)_{k+1}$ in $\bigcup_{i \ge I} (\mathcal{X}_i \cup \mathcal{Y}_i)$ and the proof of the proposition is complete. □

From the above theorem, we can deduce also decay in a "neighborhood" of $\mathcal{I}^+$.

Theorem 9.5. *Let $\epsilon > 0$ and define ω as in the previous theorem. Then there exists a constant $\tilde{C}_\epsilon > 0$ such that in $r \ge \delta_{\omega - \epsilon} v$, we have the bounds*

$$|\theta| \le \tilde{C}_\epsilon v^{-1} u^{-\omega + 1 + \epsilon}$$

$$|\zeta| \le \tilde{C}_\epsilon u^{-\omega + \epsilon}$$

$$|r\phi| \le \tilde{C}_\epsilon u^{-\omega + 1}.$$

Proof. The proof is similar to Theorem 9.2. □

10 Unanswered questions

In view of Theorem 2 of [24], we have, in addition to Theorem 1.1 of the Introduction, the following complementary result:

Theorem 10.1. *Under the assumptions of Corollary 1.1.2, if* (5) *does not hold, and if there exists a* $V > 0$ *such that, for* $v \geq V$,

$$|\theta|(\infty, v) \geq cv^{-p}, \tag{183}$$

for some $p < 9$, $c > 0$, *then the Cauchy development of initial data is* inextendible *as a* C^1 *metric.*

Heuristic analysis [28, 37, 1, 2] and numerical studies [29, 8, 33] seem to indicate that the inequality (183) holds with $p = 3$ for generic compactly supported data. This remains, however, an open problem.

On the other hand, it seems that the red-shift techniques of this paper could yield the following

Conjecture 10.1. *Given any* $\epsilon > 0$, *and* $\omega > 1$, *there exist characteristic asymptotically flat initial data, on which* $|\phi|(0, v) \leq Cr^{-1}$, $|\theta+\phi\lambda|(0, v) \leq Cr^{-\omega}$, *and for which*

$$|\phi(\infty, v)| \leq Cv^{-\omega-\epsilon},$$

for some constant $C > 0$.

A proof of the above conjecture might proceed by prescribing initial data on the event horizon and a conjugate constant-v ray, and solving this characteristic initial value problem. Note that this problem is well-posed in spherical symmetry, as the spacelike and timelike directions can be interchanged.

Conjecture 10.1 would say that the decay rate of Theorem 9.3 is sharp (modulo an ϵ) in the context of the characteristic initial value problem formulated in Section 3. At the same time, it would indicate that any argument guaranteeing (183) will have to come to terms with whatever is special about "generic" compactly supported data.

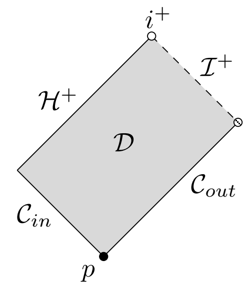
11 The uncoupled case

In this section, we will adapt the approach of this paper to the problem of the wave equation on a fixed Schwarzschild or–more generally–a non-extreme Reissner-Nordström background.[36] As we shall see, this uncoupled problem is basically described by replacing the right-hand side of (26), (27), and (28) with 0. While at first glance, it might seem that this only makes things simpler, in fact, we must be careful! Whereas the right-hand side of (28) is always handled as an error term, and thus, indeed, things are easier when it vanishes, the right-hand side of equations (26) and (27) provide the energy estimate for ϕ, fundamental to our argument! It turns out, however, that since we now have a *static* background, we can retrieve the very same energy estimate as before, not from the Einstein equations, but from Noether's theorem. To faciliate the comparison, we will actually redefine ϖ in such a way so that the equations (26)

[36] The reader can formulate for himself the most general assumptions on the background metric for which the the argument applies.

and (27) still hold precisely as before. The applicability of our argument to the uncoupled case will then be clear.

Now for the details. Let $(\mathcal{M}, g)$ be the maximally analytic non-extreme Reissner-Nordström spacetime of [30], with parameters $0 \leq |e| < M$. Note that this metric arises as a spherical symmetric solution of the system (1)–(4), if $F_{\mu\nu}$ is appropriately defined, and if ϕ is defined to vanish identically. Let $\mathcal{Q}$ denote the Lorentzian quotient of $(\mathcal{M}, g)$, let π_1 denote the natural projection $\pi_1 : \mathcal{M} \to \mathcal{Q}$, let $r : \mathcal{Q} \to \mathbf{R}$ be the area radius function, and let $\mathcal{D} \subset \mathcal{Q}$ be a subset with Penrose diagram as depicted below:



Let m be the Hawking mass, as defined in Section 2, let (u, v) be a regular coordinate system covering $\mathcal{D}$, defined as in Section 4, and let Ω, ν, λ, κ be as defined in Section 2. In particular, we have $\nu < 0$, $\kappa > 0$. The equations (26)–(28) are satisfied with ζ and θ vanishing. In particular, the renormalized Hawking mass is a constant that can easily be seen to equal M, since, at infinity, the renormalized Hawking mass tends to the Bondi mass. In what follows, we shall always denote the renormalized Hawking mass by M, as we will reserve the symbol ϖ for a new quantity to be defined shortly. We have thus that

$$\partial_u \kappa = 0, \tag{184}$$

$$\partial_u \lambda = \frac{2\nu\kappa}{r^2}\left(M - \frac{e^2}{r}\right), \tag{185}$$

and similarly

$$\partial_v \frac{1-\mu}{\nu} = 0, \tag{186}$$

$$\partial_v \nu = \frac{2\nu\kappa}{r^2}\left(M - \frac{e^2}{r}\right). \tag{187}$$

From (184) and (186), it is easy to see that

$$K = -\frac{1}{2}\frac{1-\mu}{\nu}\frac{\partial}{\partial u} + \frac{1}{2}\kappa^{-1}\frac{\partial}{\partial v}$$

defines a Killing vector on $\pi_1^{-1}(\mathcal{D})$.

Let ϕ now be a C^2 solution to the wave equation $\Box_g\phi = 0$ on $\pi_1^{-1}(\mathcal{D})$. Define the zeroth spherical harmonic $\phi_0 : \mathcal{D} \to \mathbf{R}$ by

$$\phi_0(p) = \frac{1}{4\pi r^2}\int_{\pi_1^{-1}(p)} \phi.$$

It is clear that $\pi_1^*\phi_0$ satisfies $\Box_g \pi_1^*\phi_0 = 0$.

We proceed to compute the energy identity for ϕ_0 arising from Noether's theorem applied to the Killing field K. First, we compute the components of the energy-momentum tensor $T_{\mu\nu}$ of $\pi_1^*\phi_0$. This can be written

$$T_{\mu\nu}dx^\mu\otimes dx^\nu = T_{vv}dv\otimes dv + T_{uv}(du\otimes dv + dv\otimes du) + T_{uu}du\otimes du + T_{AB}dx^A\otimes dx^B$$

where x^A denote local coordinates on the sphere. Defining $\theta = r\partial_v\phi_0$, $\zeta = r\partial_u\phi_0$, we compute

$$T_{uv} = 0,$$
$$T_{uu} = r^{-2}\zeta^2,$$
$$T_{vv} = r^{-2}\theta^2,$$
$$T_{AB} = -g_{AB}2g^{uv}\partial_u\phi\partial_v\phi = -g_{AB}\kappa^{-1}\nu^{-1}\frac{\theta\zeta}{r^2}.$$

Defining now $P^\mu = g^{\mu\nu}T_{\nu\gamma}K^\gamma$, and $\bar{P} = \pi_{1*}P$, we have

$$g\left(P, \frac{\partial}{\partial u}\right) = \bar{g}\left(\bar{P}, \frac{\partial}{\partial u}\right) = -\frac{1}{2}\frac{1-\mu}{\nu}T_{uu} = -\frac{1}{2}\frac{1-\mu}{\nu}r^{-2}\zeta^2,$$

$$g\left(P, \frac{\partial}{\partial v}\right) = \bar{g}\left(\bar{P}, \frac{\partial}{\partial v}\right) = \frac{1}{2}\kappa^{-1}T_{vv} = \frac{1}{2}\kappa^{-1}r^{-2}\theta^2.$$

Given an arbitrary characteristic rectangle $\mathcal{R} = [u_1, u_2]\times[v_1, v_2] \subset \mathcal{D}$, integrating the equation

$$\nabla\cdot P = 0$$

in $\pi_1^{-1}(\mathcal{R})$, using the divergence theorem, we obtain

$$\int_{u_1}^{u_2} 4\pi r^2\bar{g}\left(\bar{P}, \frac{\partial}{\partial u}\right)(u, v_1)du + \int_{v_1}^{v_2} 4\pi r^2\bar{g}\left(\bar{P}, \frac{\partial}{\partial v}\right)(u_1, v)dv =$$
$$\int_{u_1}^{u_2} 4\pi r^2\bar{g}\left(\bar{P}, \frac{\partial}{\partial u}\right)(u, v_2)du + \int_{v_1}^{v_2} 4\pi r^2\bar{g}\left(\bar{P}, \frac{\partial}{\partial v}\right)(u_2, v)dv,$$

and consequently

$$\int_{u_1}^{u_2}\frac{1}{2}\zeta^2\frac{1-\mu}{(-\nu)}(u, v_1)du + \int_{v_1}^{v_2}\frac{1}{2}\theta^2\kappa^{-1}(u_1, v)dv =$$
$$\int_{u_1}^{u_2}\frac{1}{2}\zeta^2\frac{1-\mu}{(-\nu)}(u, v_2)du + \int_{v_1}^{v_2}\frac{1}{2}\theta^2\kappa^{-1}(u_2, v)dv. \tag{188}$$

We have thus obtained precisely the same energy identity which was so crucial in the coupled case!

So as to facilitate comparison with the notation of this paper, we first recast (188) in differential from: Defining the 1-form η by

$$\eta = \frac{1}{2}(1-\mu)\left(\frac{\zeta}{\nu}\right)^2 \nu du + \frac{1}{2}(1-\mu)\left(\frac{\theta}{\lambda}\right)^2 \lambda dv,$$

(188) is equivalent to the statement that η is closed:

$$d\eta = 0.$$

Thus, since $\mathcal{D}$ is simply connected, it follows that, if ϕ_0 has finite energy initially, there exists a unique 0-form ϖ, such that

$$\varpi(1,\infty) = M, \tag{189}$$

in the obvious limiting sense, and

$$d\varpi = \eta, \tag{190}$$

i.e. such that ϖ satisfies equations (26) and (27).

Thus, we can consider the "uncoupled problem" as described by the system (26), (27), (184), (29), (30), (31)[37]. Note that from the above equations, one derives as before the equations

$$\partial_v\left(\frac{\zeta}{\nu}\right) = -\frac{\theta}{r} - \frac{\zeta}{\nu}\left(\frac{2\kappa}{r^2}\left(M - \frac{e^2}{r}\right)\right), \tag{191}$$

$$\partial_u(\phi\lambda + \theta) = \frac{2\kappa\nu\phi}{r^2}\left(M - \frac{e^2}{r}\right), \tag{192}$$

taking the place of (51) and (50).

Examining the propositions of this paper, one immediately sees that the estimates employed for the expression

$$\varpi - \frac{e^2}{r}, \tag{193}$$

when integrating (32) or (51), also hold for

$$M - \frac{e^2}{r}, \tag{194}$$

while the estimates for

$$\frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2 \nu, \tag{195}$$

[37]We include this equation in our system since we can no longer recover λ from κ, ϖ and r.

when integrating (28), also hold (trivially) for 0.[38] Thus, this paper can be read *mutatis mutandis* for the uncoupled case, replacing the expression (193) by (195) when integrating (32) or (51), and (194) by 0, when integrating (28). There is really only one point in the paper where the above leads to an essential simplification for the uncoupled case: In the proof of Proposition 5.2, one can immediately derive equation (73) and continue from there.

The authors hope that besides aiding the reader primarily interested in the linearized uncoupled problem, this section at the same time convinces him that the non-linear coupled problem is not much more difficult once the linear problem is looked at in a more robust setting.

A The initial value problem in the large

A.1 Basic Lorentzian geometry

We review here certain basic Lorentzian geometric concepts and notations, that will be used in the sequel without comment. The reader unfamiliar with these notions should consult a basic textbook, for instance, Hawking and Ellis [30].

An n-dimensional *spacetime* $(\mathcal{M}, g)$ is an n-dimensional C^2 Hausdorff manifold $\mathcal{M}$ together with a time-oriented C^1 Lorentzian[39] metric g. Tangent vectors are called *timelike*, *spacelike*, or *null*, according to whether their length is negative, positive, or zero, respectively. Timelike and null vectors collectively are known as *causal*. A time-orientation is defined by a continuous timelike vector field T defined throughout $\mathcal{M}$. Causal vectors are called *future directed* if their inner product with T is negative, and *past directed* if their inner product is positive. A parametrized C^1-curve γ in $\mathcal{M}$ is said to be timelike, spacelike, or null, according to whether its tangent vector has negative, positive, or zero length, everywhere. If $\dot{\gamma}$ is everywhere causal, γ is called causal, and if $\dot{\gamma}$ is future directed, γ is called future directed. An affine parametrization on a parametrized geodesic γ is one such that $\frac{\partial}{\partial s}$ is parallel along γ.

For a subset $S \subset \mathcal{M}$, we define the *causal future* of S, denoted by $J^+(S)$, to be the set of points which can be reached from S by a future pointing causal curve. We define the *causal past of* $J^-(S)$ to be the set of points which can be reached from S by a past pointing causal curve. The chronological future $I^+(S)$ is defined similarly, where "causal" is replaced by "timelike". A subset $S \subset \mathcal{M}$ is said to be *achronal* if $S \cap I^+(S) = \emptyset$.

A submanifold of $\mathcal{M}$ is called *spacelike* if its induced metric is Riemannian, *null* if its induced metric is degenerate, and timelike if its induced metric is *Lorentzian*. The reader can check that for embedded curves, this definition coincides with the previous. Also, one can easily see that a hypersurface is spacelike, timelike, or null if and only if its normal is everywhere timelike, spacelike, or null, respectively.

[38] We have chosen the normalization (189) precisely so that (37) holds.

[39] Our convention will be signature $(-1, 1, \overset{n-1}{\ldots}, 1)$.

A curve is called *inextendible* if it is not a proper subset of another curve. A spacetime such that all inextendible future directed causal geodesics extend for arbitrary large positive values of an affine parameter are called *future causally geodesically complete*, otherwise *future causally geodesically incomplete*. A spacelike hypersurface $\mathcal{S}$ is said to be *Cauchy* if every inextendible parametrized causal curve in $\mathcal{M}$ intersects $\mathcal{S}$ once and only once. A spacetime possessing a Cauchy surface is said to be *globally hyperbolic*.

A.2 The maximal Cauchy development

For convenience, in the rest of this section all functions, tensors, maps and actions referred to are by assumption C^∞.

We say that a 4-dimensional spacetime $(\mathcal{M}, g)$, together with an anti-symmetric 2-tensor $F_{\mu\nu}$ and a function ϕ satisfy the Einstein-Maxwell-scalar field equations if

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= 2T_{\mu\nu}, \\
F^{;\mu}_{\mu\nu} = 0, F_{[\mu\nu,\lambda]} &= 0, \\
g^{\mu\nu}\phi_{;\mu\nu} &= 0, \\
T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi^{;\alpha}\phi_{;\alpha} &+ F_{\mu\alpha}F^{\alpha}_{\ \nu} - \frac{1}{4}g_{\mu\nu}F^{\alpha}_{\ \lambda}F^{\lambda}_{\ \alpha}.
\end{aligned}
$$

Here $R_{\mu\nu}$ denotes the Ricci curvature of $g_{\mu\nu}$, and R the scalar curvature. We call a connected Riemannian 3-manifold $(\Sigma, \overset{\circ}{g})$, together with a 2-symmetric tensor K_{ij}, two one forms E_i, B_i, and functions $\overset{\circ}{\phi}$, ϕ' initial data for the Einstein-Maxwell-scalar field, equations if the following constraint equations are satisfied:

$$
\overset{\circ}{R} - K_{ij}K^{ij} + (K^i_i)^2 = (\phi')^2 + \overset{\circ}{\phi}{}^{;i}\overset{\circ}{\phi}_{;i} + E^iE_i + B^iB_i,
$$

$$
K^{;j}_{ij} - K^{j}_{j;i} = \phi' \overset{\circ}{\phi}_i + 2\epsilon_{ijk}E^jB^k,
$$

where $\overset{\circ}{R}$ is the scalar curvature of the metric $\overset{\circ}{g}$ on Σ, ϵ_{ijk} is the fully anti-symmetric tensor representing its volume form, and raised, covariant, etc. latin indices are with respect to $\overset{\circ}{g}$ and its Levi-Civita connection $\overset{\circ}{\nabla}$.

The theorem of Choquet-Bruhat and Geroch [12], applied to the above equations, yields

Theorem A.1. *Given an initial data set $\{\Sigma, \overset{\circ}{g}_{ij}, K_{ij}, E_i, B_i, \overset{\circ}{\phi}, \phi'\}$, there exists a unique solution $\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$ to the Einstein-Maxwell-scalar field equations such that*

1. *$(\mathcal{M}, g)$ is a globally hyperbolic spacetime.*

2. *Σ is a submanifold of $(\mathcal{M}, g)$ with first fundamental form $\overset{\circ}{g}_{ij}$ and second fundamental form K_{ij}, with respect to a future-pointing unit normal N. Moreover, $\phi|_{\mathcal{S}} = \overset{\circ}{\phi}$, $N^\alpha\phi_{;\alpha} = \phi'$, $N^\alpha F_{\alpha i} = E_i$, $-\frac{1}{2}N^\alpha\epsilon_{\alpha i}{}^{\gamma\delta}F_{\gamma\delta} = B_i$.*

3. *Given a second solution $\{\tilde{\mathcal{M}}, \tilde{g}, \tilde{F}_{\mu\nu}, \tilde{\phi}\}$ such that the above two properties holds, there exists a map*
$$\Phi : \tilde{\mathcal{M}} \to \mathcal{M}$$
such that $\Phi^* g = \tilde{g}$, $\Phi^* F = \tilde{F}$, $\Phi^* \phi = \tilde{\phi}$, where Φ^* denotes the induced pull back map.

The above collection $\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$, is known as the *Cauchy development* or the *maximal domain of development* of $\{\Sigma, \overset{\circ}{g}, K, E_i, B_i, \overset{\circ}{\phi}, \phi'\}$. The set $\mathcal{M} \cap J^+(\Sigma)$,[40] together with the restriction of $F_{\mu\nu}$, ϕ to this set, is the *future Cauchy development*.

A.3 Penrose's singularity theorem

If $\mathcal{E} \subset \mathcal{M}$ is a closed spacelike 2-surface, then the intersection of a tubular neighborhood of $\mathcal{E}$ with $J^+(\mathcal{E}) \setminus (I^+(\mathcal{E}) \cup \mathcal{E})$ is a null 3-surface with 2 connected components. $\mathcal{E}$ can be considered as a smooth boundary of either; if with respect to future-pointing normals, the mean curvature of $\mathcal{E}$ is everywhere negative for both connected components, then we say $\mathcal{E}$ is *trapped*. Penrose's celebrated singularity theorem is given by

Theorem A.2. *Consider a globally hyperbolic spacetime $(\mathcal{M}, g)$ with a non-compact Cauchy surface Σ. Suppose for all null vectors L, the Ricci curvature of g satisfies the inequality $Ric(L, L) \geq 0$. Then, it follows that if $\mathcal{M}$ contains a closed trapped surface $\mathcal{E}$, then $(\mathcal{M}, g)$ is future-causally geodesically incomplete.*

It is easy to check that the assumptions of the theorem are satisfied by the future Cauchy development of non-compact initial data for the Einstein-Maxwell-scalar field system, if it is assumed that the data contain a trapped surface $\mathcal{E}$.

A.4 Asymptotically flat data

The study of isolated self-gravitating systems leads to the idealization of so-called asymptotically flat initial data. Let $\{\Sigma, \overset{\circ}{g}, K, E_i, B_i, \overset{\circ}{\phi}, \phi'\}$ be a data set, and let $\mathcal{U} \subset \Sigma$ be open. $\mathcal{U}$ is said to be an *asymptotically flat end* if $\mathcal{U}$ is diffeomorphic to $\mathbf{R}^3 \setminus B_1(0)$, and if with respect to the coordinate chart induced by this diffeomorphism, the following are satisfied:

$$\left| g_{ij}(x) - \left(1 + \frac{M}{|x|}\right) \delta_{ij}(x) \right| + |x||K_{ij}(x)| \leq C|x|^{-1-\alpha}$$

$$|\phi'(x)| + |\nabla \overset{\circ}{\phi}| \leq C|x|^{-2-\alpha}$$

$$\left| E_i(x) - \frac{e}{4\pi} \frac{x_i}{|x|^3} \right| + |H_i(x)| \leq C|x|^{-2-\alpha} \tag{196}$$

[40] By our convention, this is a manifold with (spacelike) boundary.

where $|\cdot|$ denotes the Euclidean metric, constant $\alpha > 0$ and M and e are the mass and the charge respectively. The data set is said to be *asymptotically flat with k ends* if there exists a compact set $\mathcal{K} \subset \Sigma$ such that

$$\Sigma \setminus \mathcal{K} = \bigcup_{i=1}^{k} \mathcal{U}_k$$

where the above union is disjoint, and each $\mathcal{U}_k$ is an asymptotically flat end.

A.5 The cosmic censorship conjectures

With this, we can formulate *strong cosmic censorship*[41]:

Conjecture A.1. *For a "generic" asymptotically flat complete initial data set for the Einstein-Maxwell-scalar field system, the Cauchy development is inextendible as a manifold with C^0 metric.*

Physically speaking, strong cosmic censorship, as formulated above, ensures that observers travelling to the end of the Cauchy development are "destroyed". It is for this reason that our assumption on inextendibility be with respect to continuous metrics, and not more regular metrics. See [16].

In addition to the above conjecture, there is the so-called *weak cosmic censorship* conjecture. A definite formulation is given by

Conjecture A.2. *For a "generic" complete asymptotically flat initial data set the Einstein-Maxwell-scalar field system, the Cauchy development posseses a complete null infinity in the sense of Christodoulou [16].*

In the case where the Maxwell field identically vanishes, the above conjecture was established by Christodoulou [14], when one restricted to initial data which are spherically symmetric. If the Maxwell field does not vanish, then the completeness of null infinity for complete asymptotically flat spherically symmetric initial data follows immediately from [26]. It should be noted, however, that the fact that the initial data set has two asymptotically flat ends renders weak cosmic censorship almost trivial in this case.

B Spherical symmetry

A solution $\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$ of the Einstein-Maxwell-scalar field system (1)–(4) is said to be *spherically symmetric* if the group $SO(3)$ acts nontrivially by isometry on $(\mathcal{M}, g)$, and preserves the matter fields ϕ and $F_{\mu\nu}$, in the sense that ϕ is constant on group orbits and $F_{\mu\nu}$ is invariant with respect to the induced action of $SO(3)$ in $\Lambda^2(T\mathcal{M})$.

We will call an initial data set $\{\Sigma, \mathring{g}, K_{ij}, E_i, B_i, \mathring{\phi}, \phi'\}$ spherically symmetric if $SO(3)$ acts by isometry on $(\Sigma, \mathring{g})$, if this action preserves K, ϕ, E_i, and B_i and

[41] This formulation is due to Christodoulou [16].

if, moreover, $\mathcal{S} = \Sigma/SO(3)$ inherits the structure of a connected 1-dimensional Riemannian manifold, possibly with boundary.[42] (In particular, it is clear that Σ can have at most 2 ends.)

We have the following:

Theorem B.1. *Let $\{\Sigma, \overset{\circ}{g}, B_i, E_i, \overset{\circ}{\phi}, \phi'\}$ be an asymptotically flat spherically symmetric initial data set. Then its Cauchy development $\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$ is also spherically symmetric.*

We omit the proof. The above theorem says that our symmetry assumptions are compatible with the problem of evolution.

From the point of view of p.d.e. theory, the fundamental fact is that the problem of evolution now reduces to a system of equations on a 2-dimensional domain. The setting for this system is the quotient space $\mathcal{Q} = \mathcal{M}/SO(3)$. In fact, $\mathcal{Q}$ inherits the structure of a 2-dimensional Lorentzian manifold, possibly with boundary, with metric $\bar{g}$, and a natural function $r : \mathcal{Q} \to \mathbf{R}$, called the *area-radius*, defined by

$$r(p) = \sqrt{\frac{Area(p)}{4\pi}}. \tag{197}$$

(In particular, the $\infty > Area(p) \geq 0$ for all group orbits p; these group orbits will be spacelike spheres or points.) The metric g on $\mathcal{M}$ can then be written as

$$g = \bar{g} + r^2\gamma,$$

where γ is the standard metric on the unit 2 sphere. The function ϕ descends to a function on $\mathcal{Q}$, which, without fear of confusion, we shall also denote by ϕ. It was shown in [23] that the energy momentum tensor $T_{\mu\nu}$ can be written as

$$\begin{aligned} T_{\mu\nu}dx^\mu \otimes dx^\nu &= \frac{e^2}{2r^4}\bar{g} + \frac{e^2}{r^2}\gamma \\ &\quad + \phi_\mu\phi_\nu dx^\mu \otimes dx^\nu - \frac{1}{2}g_{\mu\nu}g(\nabla\phi, \nabla\phi)dx^\mu \otimes dx^\nu \end{aligned} \tag{198}$$

where e is the constant of (196). Thus (2) decouples and the evolution of g and ϕ can be determined by a second order system of equations for $\bar{g}$, r, ϕ on $\mathcal{Q}$. Introducing null coordinates, and defining quantities as in Assumptions $\mathbf{\Gamma}'$–$\mathbf{E}'$, then it was shown in [23] that this system can be written as (26)–(30).

Let Γ denote the subset of $\mathcal{Q}$ such that $r = 0$. If nonempty, Γ is a connected timelike boundary for $\mathcal{Q}$, intersecting $\Sigma/SO(3)$. Γ is called the *axis of symmetry* and corresponds precisely to the fixed points of the $SO(3)$ action. By (198), it is clear that regularity requires that if Γ is non-empty, then $e = 0$. Now, if $(\Sigma, \overset{\circ}{g})$ is assumed in addition to be *complete*, then Σ has one end if Γ is non-empty, and two ends if Γ is empty. Thus, if $e \neq 0$, in view of the above remarks, it follows that a complete Σ necessarily has *two* asymptotically flat ends.

[42] We require, in other words, Σ to be a warped product of a connected subset of $\mathcal{R}$ with S^2, where $SO(3)$ acts transitively on the S^2 factor. It is this that ensures that $\mathcal{Q}$ described in what follows is indeed a manifold (with boundary).

C Penrose diagrams

In the introduction to this paper, we have used so-called *Penrose diagram* notation to conveniently convey certain global geometric information about spherically symmetric spacetimes. Briefly put, the domain of a Penrose diagram is the image of a map $\Phi : (\mathcal{Q}, \bar{g}) \to (\mathbf{R}^{1+1}, -dudv)$ into 2-dimensional Minkowski space, such that Φ preserves the causal structure, and is bounded. (Alternatively, one can think of the diagram as the domain of a bounded global null coordinate system in the (u, v) plane.) Since one can determine the causal relations of two points in a bounded subset of Minkowski space by inspection, the causal geometry of $\mathcal{Q}$ can be easily perceived by displaying its image under Φ.

There is more, however, to Penrose diagrams than this purely causal aspect. The map Φ defines a boundary of $\mathcal{Q}$, namely $\partial(\Phi(\mathcal{Q})) \subset \mathbf{R}^2$. The natural function $r : \mathcal{Q} \to \mathbf{R}$ defined in Appendix B allows one to distinguish special subsets of this boundary, in particular, a set to be called *null infinity*, denoted $\mathcal{I}^+$. Causal concepts extend to $\overline{\mathcal{Q}}$. With this, one will be able to define the black hole region $\mathcal{Q} \setminus J^-(\mathcal{I}^+)$ and the event horizon $(\partial J^-(\mathcal{I}^+)) \cap \mathcal{Q}$.

We proceed in the following sections to discuss these issues in more detail. We begin in Section C.1 with a review of the geometry of 2-dimensional Minkowski space. In Section C.2, we define the concept of a Penrose diagram, and the induced concepts of black hole and event horizon. In Section C.3, we illustrate certain conventions on incidence relations when displaying domains of $\mathbf{R}^2$, and finally, we give the Penrose diagrams of some celebrated special solutions of the Einstein equations, in Section C.4.

C.1 2-dimensional Minkowski space

We call the spacetime $(\mathbf{R}^2, -dt^2 + dx^2)$ 2-*dimensional Minkowski space*. It is time-oriented by the vector $\frac{\partial}{\partial t}$. To examine the causal properties, it is more convenient to introduce null coordinates $u = t - x$, $v = t + x$. One easily sees that a vector $a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}$ is causal if $ab \geq 0$, in which case it is future pointing if $a > 0$, timelike if $ab > 0$, and null if $ab = 0$. Thus

$$J^+(u_0, v_0) = \{(u, v) | u \geq u_0, v \geq v_0\},$$

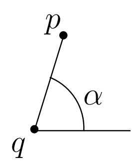
$$J^-(x_0, t_0) = \{(u, v) | u \leq u_0, v \leq v_0\},$$

$$I^+(u_0, v_0) = \{(u, v) | u > u_0, v > v_0\},$$

$$I^-(x_0, t_0) = \{(u, v) | u < u_0, v < v_0\}.$$

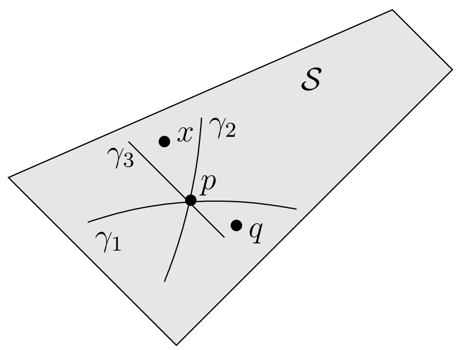
If the original x and t coordinates are mapped to the Euclidean plane as horizontal and vertical, then the constant-u and constant-v lines are at 45 degree, and 135 degree angles from the horizontal, respectively. Thus $p \in J^+(q)$ if the

angle α between p and q, as measured from the horizontal:



satisfies $45 \leq \alpha \leq 135$. Similarly $p \in J^-(q)$ if $-45 \geq \alpha \geq -135$ degrees. Null curves are precisely those of constant u or v, i.e. line segments at 45 and 135 degrees, whereas timelike curves have tangent with angle from the horizontal α satisfying $45 < \alpha < 135$ or $-45 > \alpha > -135$.

As an exercise, the reader can read off from below



the following incidence and causal relations: $p, q, x \in \mathcal{S}$, $\gamma_1, \gamma_2, \gamma_3 \subset S$, $q \notin J^-(p) \cup J^+(p)$, $x \in I^+(p)$, $p = \gamma_1 \cap \gamma_2 \cap \gamma_3$, and that γ_1, γ_2, and γ_3 are spacelike, timelike, and null, respectively.

C.2 Penrose diagrams

We will define a *Penrose diagram* to consist of the following information:

1. $(\mathcal{Q}, \bar{g})$ a 2-dimensional C^2 manifold, possibly with boundary, with a time-oriented C^1 Lorentzian metric.

2. A C^2 diffeomorphism preserving the causal structure:

$$\Phi : \mathcal{Q} \to \mathcal{PD}$$

where $\mathcal{PD} \subset \mathbf{R}^2$ is a bounded region.

3. A continuous nonnegative function $r : \mathcal{Q} \to \mathbf{R}$.

The domain of the Penrose diagram is $\mathcal{PD}$; this set and its closure inherit a topology and causal structure from $\mathbf{R}^{1+1}$.

For Cauchy developments of spherically symmetric asymptotically flat initial data for a wide variety of Einstein-matter systems, the Lorentzian quotient

$(\mathcal{Q}, g)$, together with the area radius function r, will indeed admit a Penrose diagram, in view of the existence of a globally defined null coordinate system (u, v), which can be chosen in addition to be bounded. The coordinate functions together determine the map Φ.

In general, given a Penrose diagram $\{(\mathcal{Q}, g), r, \Phi\}$, we can define a subset

$$\mathcal{I}^+ \subset \partial\mathcal{PD} = \overline{\mathcal{PD}} \setminus \mathcal{PD}$$

as follows:

$$\mathcal{I}^+ = \{p \in \partial\mathcal{PD} : \forall_{R,\epsilon>0}, \exists_{q\in(J^-(p)\setminus I^-(p))\cap B_\epsilon(p)\cap\mathcal{PD}} : r(q) \geq R\}$$

For $\mathcal{Q}$ which arises from the Cauchy development of spherically symmetric asymptotically flat data as above, a lot can be said about $\mathcal{I}^+$. In particular, if $\mathcal{Q}$ has one end, then $\mathcal{I}^+$ will be a connected null curve, with past limit point common with the limit point of the quotient of the Cauchy surface. (This common limit point is often called *spacelike infinity*, and denoted i^0.) In this case we call $\mathcal{I}^+$ *future null infinity.*[43] See [26].

With this definition of null infinity, we can define the black hole as the region

$$\mathcal{Q} \setminus \Phi^{-1}(J^-(\mathcal{I}^+))$$

and the *future event horizon*, denoted $\mathcal{H}^+$, by

$$\mathcal{H}^+ = \partial(\mathcal{Q} \setminus \Phi^{-1}(J^-(\mathcal{I}^+))) \cap \mathcal{Q}.$$

Since the black hole is a future subset of $\mathcal{Q}$, $\mathcal{H}^+$ is in fact its past boundary. In the context of the spacetimes considered here, it is easy to see that these definitions do not depend on Φ. Similarly, we can define white holes and $\mathcal{H}^-$, by interchanging future and past.

Finally, we note that Penrose diagrams are intimately related to null coordinate systems. Indeed, given a Penrose diagram, the pull back of the standard u and v coordinates on $\mathbf{R}^2$ define null coordinates on $\mathcal{Q}$, and similarly, given $\{(\mathcal{Q}, g), r\}$ and a null coordinate system (u, v) with bounded range satisfying $J^+(u_0, v_0) = \{(u, v)|u \geq u_0, v \geq v_0\} \cap \mathcal{Q}$, $J^-(u_0, v_0) = \{(u, v)|u \leq u_0, v \leq v_0\} \cap \mathcal{Q}$, these coordinates determine a map to $\mathbf{R}^2$ defining a Penrose diagram.

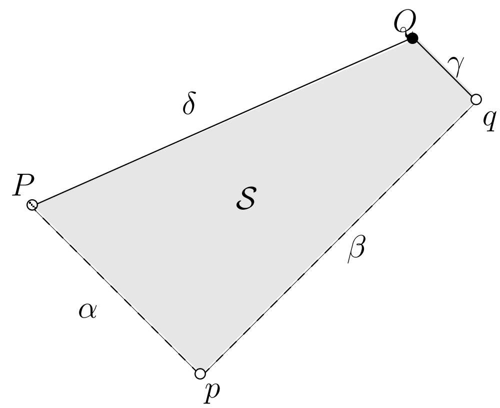
C.3 Conventions

In displaying subsets of Minkowski space, in particular Penrose diagrams, it will be useful to have certain conventions on inclusion and incidence relations.

Let us adopt the following then: Curves depicted by dashed lines are assumed not to be contained in shaded regions that they bound, and points depicted by white circles are assumed not to be contained in either the shaded regions or the

[43] If $\mathcal{Q}$ has 2 ends, then $\mathcal{I}^+$ will have 2-connected components of this form.

lines to which they are adjacent. Finally, circles filled in with lines are included in the adjacent line segment to which the filling is parallel. Example: from

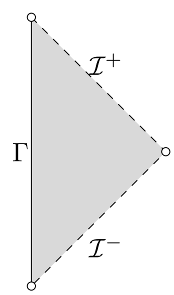


we are to understand all of the following: $p, P, q, Q, \in \overline{\mathcal{S}}$, $\alpha, \beta, \gamma, \delta \subset \overline{\mathcal{S}}$, $\mathcal{S} \subset I^+(p)$, $\alpha, \beta \subset J^+(p) \setminus I^+(p)$, $\gamma \subset J^+(q) \setminus I^+(q)$, $J^+(q) \cap \mathcal{S} = \gamma$, $\overline{\alpha} \cap \overline{\beta} = \{p\}$, but $p \notin \alpha \cup \beta$, $\delta \subset \mathcal{S}$ is spacelike, $P \notin \delta$, but $P \in \alpha$, $Q \in \delta$. Finally, dotted (as opposed to dashed) lines with arrows are reserved to help label sets.

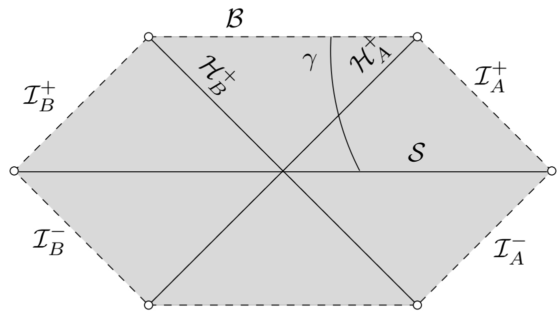
Also, it goes without saying that when displaying a Penrose diagram, in view of the fact that $\Phi : \mathcal{Q} \to \mathcal{PD}$ is $1-1$, we can use the same symbol for a subset of $\mathcal{Q}$ and its image in $\mathcal{PD}$.

C.4 Examples

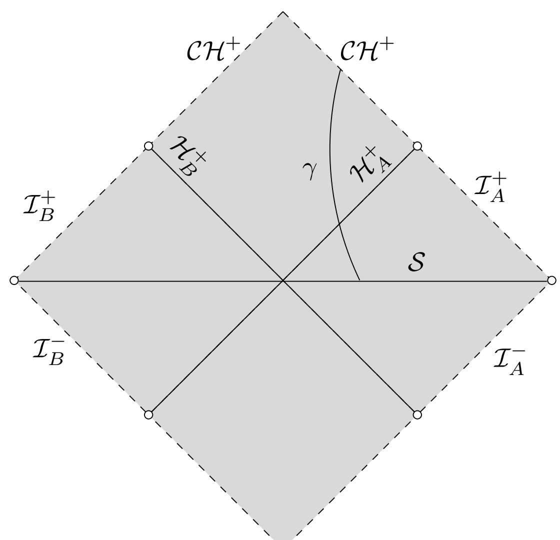
We provide for reference the Penrose diagrams of $3+1$ dimensional Minkowski space:



the Schwarzschild solution:



and the Reissner-Nordström solution[44]:



Note that the future-inextendible timelike geodesic depicted by γ has finite length in the above two spacetimes. All three spacetimes are spherically symmetric solutions of (1)–(4).

[44]We mean here the Cauchy development of the complete time symmetric Cauchy surface Σ, whose quotient $\mathcal{S} = \Sigma/SO(3)$ is depicted, *not* a so-called maximally analytic extension.

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